

Maximum-Sized Golden-Mean Matroids

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Matroids – Definitions

- A **matroid**, (E, \mathcal{I}) , consists of a finite set (E) and a collection of subsets (\mathcal{I}) . \mathcal{I} obeys certain axioms.
- The subsets in \mathcal{I} are called **independent sets**.
- The cardinality of a maximum-sized independent set is the **rank** of the matroid.
- A **circuit**, C , is a subset of E such that C is not independent, but every proper subset of C is.
- A matroid is **simple** if all circuits have size at least 3.

Matroids – Representation

- One way to construct a matroid is from a matrix over a field \mathbb{F} .
- The independent sets of the matroid are linearly independent sets of columns from the matrix.
- Such a matroid is said to be representable over \mathbb{F} .

A matrix over $GF(2)$

$$\begin{array}{cccc} & a & b & c & d \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] \end{array}$$

$$\begin{aligned} E &= \{a, b, c, d\} \\ \mathcal{I} &= \{\emptyset, \{a\}, \{b\}, \{c\}, \\ &\quad \{a, c\}, \{b, c\}\} \end{aligned}$$

Matroids – Geometries

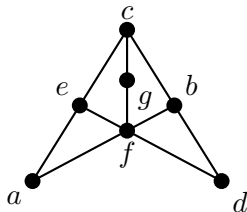
- Another way to look at simple rank 3 matroids is as geometric representations.
- A set of 3 points in the geometric representation are collinear if and only if they are not independent.

Matroids – An example

A matrix over $GF(4)$

$$\begin{array}{c} a \quad b \quad c \quad d \quad e \quad f \quad g \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & \alpha \end{array} \right] \end{array}$$

- $\{a, b, c\}$ is independent
- $\{b, c, d\}$ is a circuit



- The points a , b , and c are not collinear.
- The points b , c , and d are collinear.

Maximum-Sized

Let \mathcal{M} be a class of simple matroids, and let $M = (E_M, \mathcal{I}_M)$ be a matroid of rank r from \mathcal{M} .

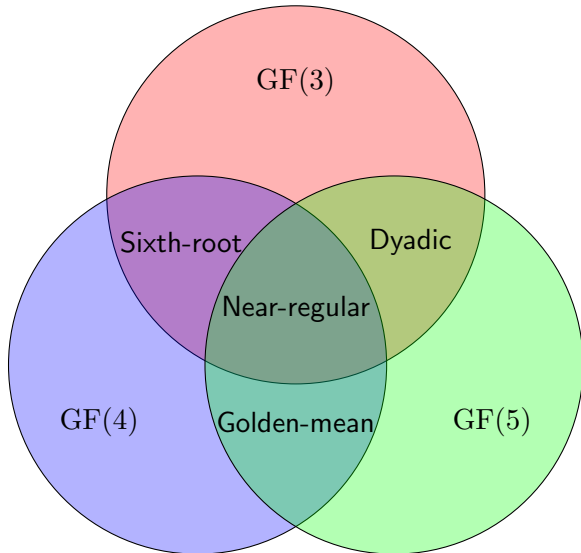
M is **maximum-sized** in \mathcal{M} if for every $N = (E_N, \mathcal{I}_N)$ in \mathcal{M} such that N has rank r , $|E_N| \leq |E_M|$.

The target

Theorem

Let M be a simple maximum-sized golden-mean matroid of rank r . Then M is “some matroid”.

Characterisation due to Vertigan and Whittle



All these classes are defined by subdeterminants falling into certain sets.

For example, the golden mean determinants are from the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$, where ϕ is the positive root of $\phi^2 - \phi - 1$.

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \phi & 1 & 1 & 0 & 0 & \phi & \phi^2 \\ 0 & 0 & 1 & 1 & \phi^2 & 1 & \phi & -\phi & 1 & 1 & \phi^2 \end{bmatrix}$$

- In this matrix, every non-zero subdeterminant is in the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$.
- For example, $\begin{vmatrix} \phi & \phi \\ \phi^2 & 1 \end{vmatrix} = \phi - \phi^3 = \phi(1 - \phi^2) = \phi(-\phi) = -\phi^2$.

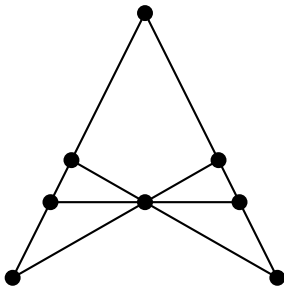
Existing Results

The maximum-sized matroids in the following classes have been completely characterised.

- Dyadic (Kung and Oxley, 1988; Kung, 1990)
- Sixth-root (Oxley, Vertigan and Whittle, 1998)
- Near-regular (Oxley, Vertigan and Whittle, 1998)

An infinite family – T

$$T_r = \left[\begin{array}{c|ccc|ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & \end{array} \begin{array}{c} I_{r-1} \\ I_{r-1} \\ I_{r-1} \\ D_{r-1} \end{array} \right]$$

T_3 

Near Regular

Theorem (Oxley, Vertigan and Whittle, 1998)

Let M be a maximum-sized near-regular simple matroid of rank r . Then M is isomorphic to T_r .

Previous Work

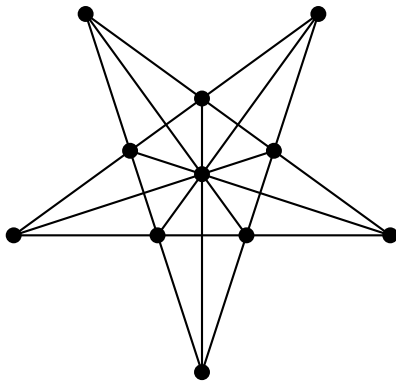
Less has been done about golden-mean matroids.

- Semple did some introductory work in his Masters thesis.
- Archer explored low rank golden-mean matroids in his PhD thesis.
- With the aid of some computer software, Archer conjectured about maximum-sized golden-mean matroids.

Progress

- Archer's conjecture is false.
- A modified version of Archer's conjecture is believed to be true.
- The existing techniques don't work on this conjecture.

Betsy Ross (B_{11})



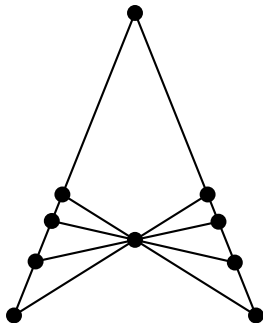
The Betsy Ross matroid, or B_{11} .

An infinite family – T^2

$$T_r^2 = \left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc} 1 & 0 & \dots & 0 & 1 & \dots & 1 & \phi & \dots & \phi & \phi^2 & \dots & \phi^2 & 0 & \dots & 0 \\ \hline 0 & & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & & & & \end{array} \right]$$

I_k I_k I_k I_k D_k

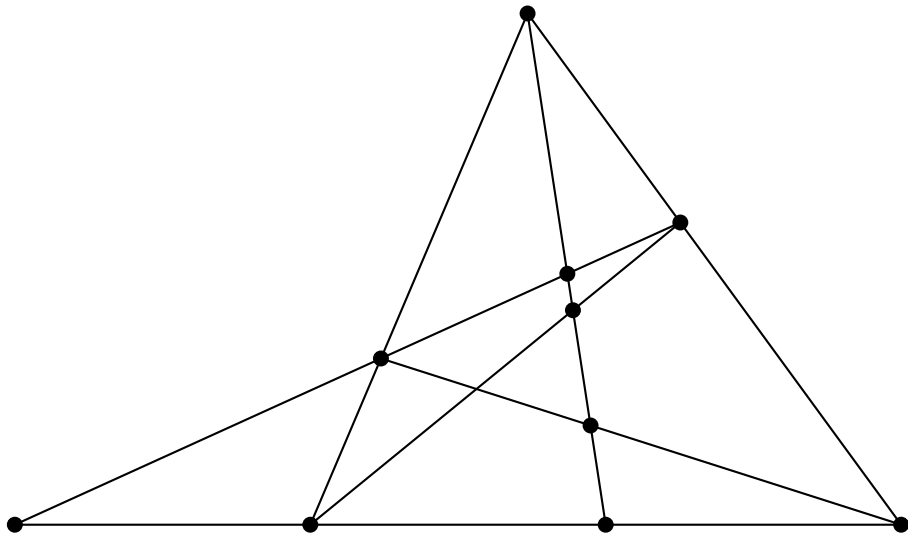
Note that $k = r - 1$.

T_3^2 

An infinite family - GI

$$GI_r = \left[\begin{array}{c|c|c|c|c|c|c} -\phi & -\phi & -\phi & 0 \dots 0 & \phi \dots \phi & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ 1 & \phi & \phi^2 & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 & \phi \dots \phi & 1 \dots 1 & 0 \dots 0 \\ \hline & 0_3^k & & I_k & I_k & I_k^0 & I_k & I_k^0 & D_k \end{array} \right]$$

Note that $k = r - 2$.

GI_3 

The Conjecture

Conjecture (Archer, 2005; Welsh, 2010)

Let M be a maximum-sized golden-mean matroid of rank r . If M has rank 3, then M is isomorphic to B_{11} , otherwise M is isomorphic to either GI_r or T_r^2 .

Minors

- There are two operations used to remove an element from a matroid:
 1. Deletion; and
 2. Contraction.
- When these operations are used, a **minor** is created.

A stepping stone

Theorem (Welsh, 2010)

Let M be a maximum-sized golden-mean matroid of rank r , with no Γ minor. Then M is isomorphic to T_r^2 .

