Maximum-Sized Golden-Mean Matroids

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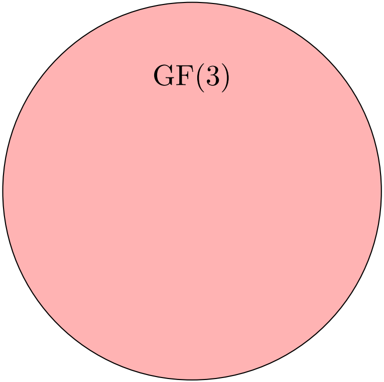
Maximum-Sized

Let $\mathcal{M}$ be a class of simple matroids, and let $M = (E_M, \mathcal{I}_M)$ be a matroid of rank r from $\mathcal{M}$.
M is **maximum-sized** in $\mathcal{M}$ if for every $N = (E_N, \mathcal{I}_N)$ in $\mathcal{M}$ such that N has rank r, $|E_N| \leq |E_M|$.

Characterisation due to Vertigan and Whittle



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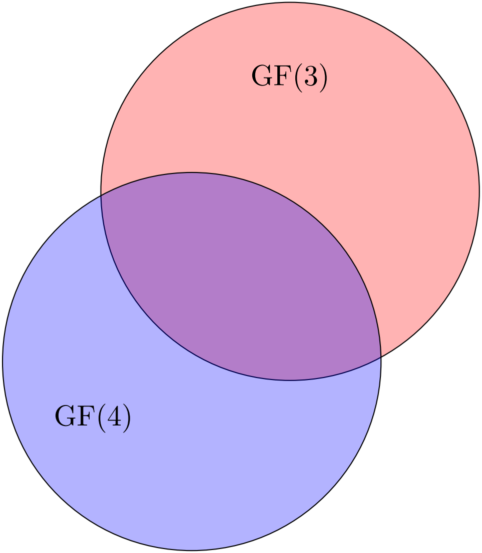

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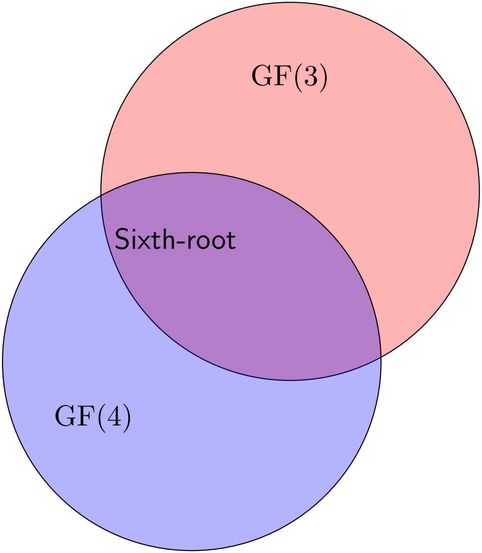

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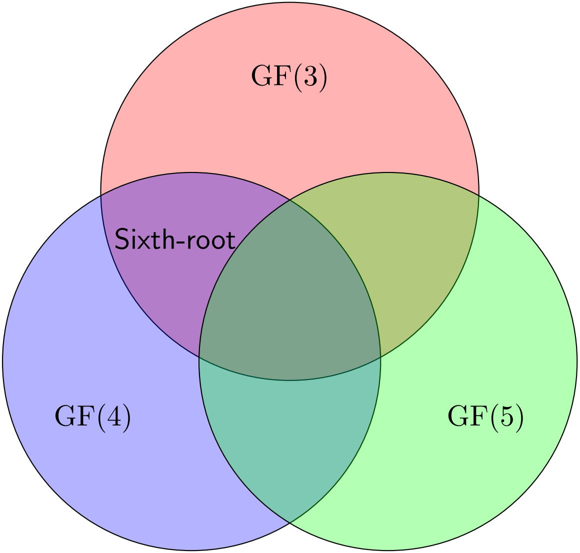

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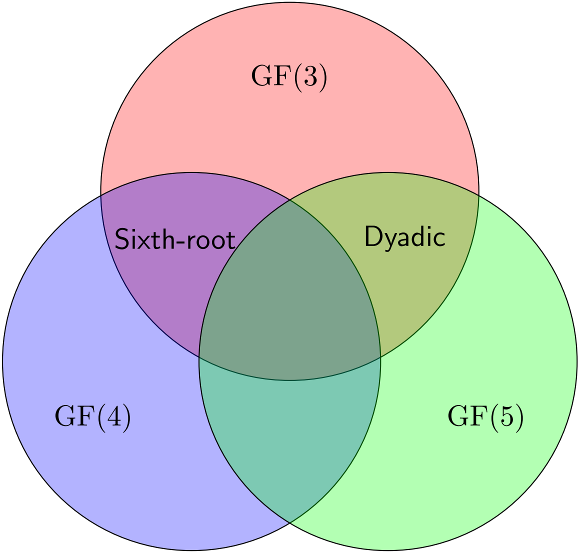

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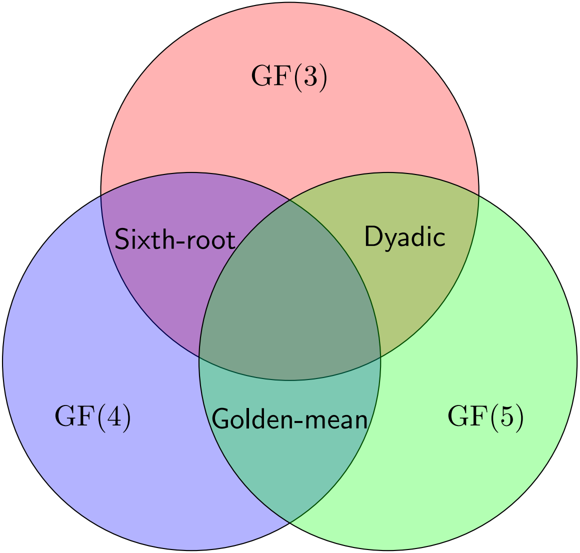

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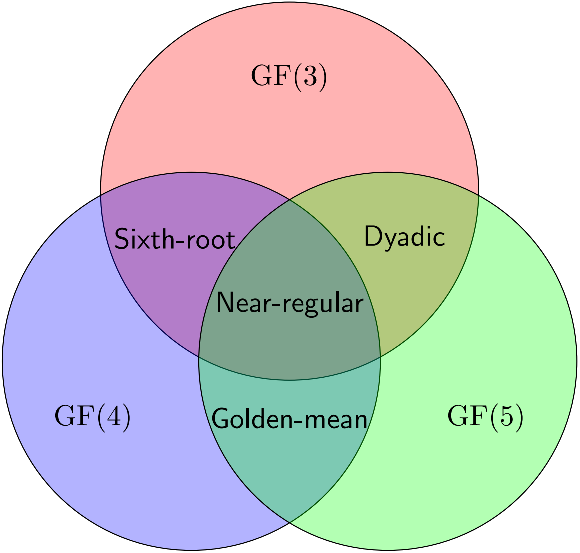

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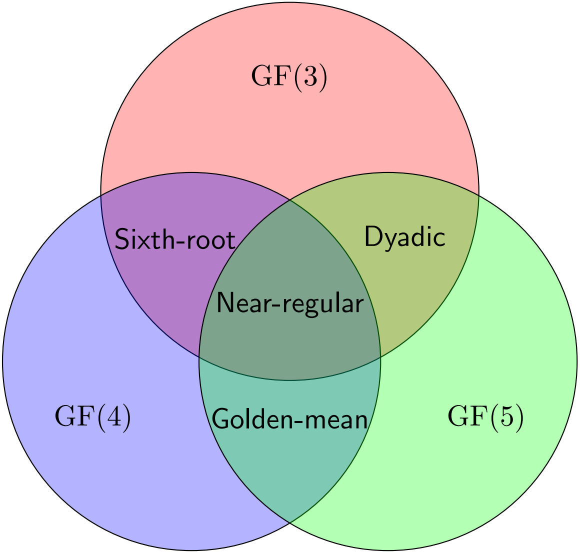

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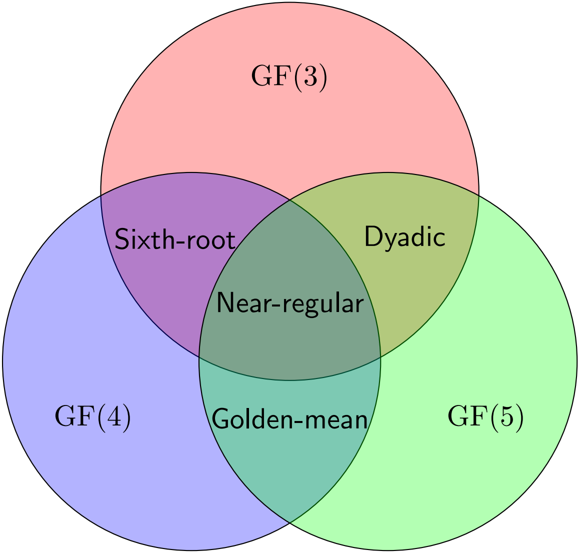

All these classes are also defined by subdeterminants falling into certain sets.
For example, the golden mean determinants are from the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$, where ϕ is the positive root of $\phi^2 - \phi - 1$.

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \phi & 1 & 1 & 0 & 0 & \phi & \phi^2 \\ 0 & 0 & 1 & 1 & \phi^2 & 1 & \phi & -\phi & 1 & 1 & \phi^2 \end{bmatrix}$$

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- In this matrix, every non-zero subdeterminant is in the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$.
- For example, $\begin{vmatrix} \phi & \phi \\ \phi^2 & 1 \end{vmatrix} = \phi - \phi^3 = \phi(1 - \phi^2) = \phi(-\phi) = -\phi^2$.

The target

Theorem

Let M be a simple maximum-sized golden-mean matroid of rank r. Then M is "some matroid".

Existing Results

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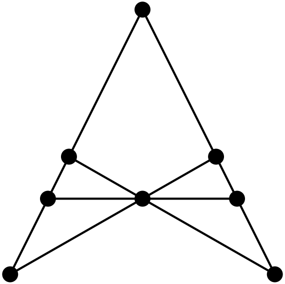
- Sixth-root (Oxley, Vertigan and Whittle, 1998)
- Near-regular (Oxley, Vertigan and Whittle, 1998)

An infinite family – T

$$T_r = \left[\begin{array}{c|ccc|ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & \\ \vdots & & I_{r-1} & & & I_{r-1} & & & I_{r-1} & & & D_{r-1} & \\ 0 & & & & & & & & & & & & \end{array}\right]$$

T_3



Near Regular

Theorem (Oxley, Vertigan and Whittle, 1998)

Let M be a maximum-sized near-regular simple matroid of rank r. Then M is isomorphic to T_r.

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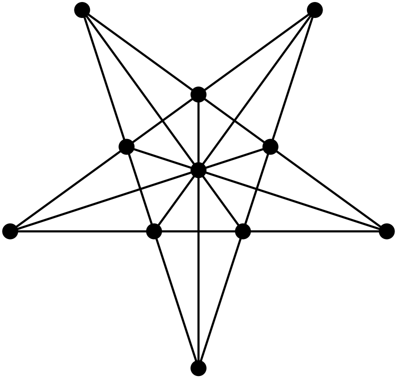
- Archer explored low rank golden-mean matroids in his PhD thesis.

Betsy Ross (B_{11})



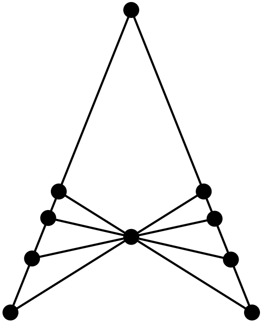
The Betsy Ross matroid, or B_{11}.

An infinite family – T^2

$$T_r^2 = \left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & \alpha^2 & \cdots & \alpha^2 & 0 & \cdots & 0 \\
\hline
0 & & & & & & & & & & & & & & & \\
\vdots & & I_k & & & I_k & & & I_k & & & I_k & & & D_k & \\
0 & & & & & & & & & & & & & & &
\end{array}\right]$$

Note that $k = r - 1$.

T_3^2

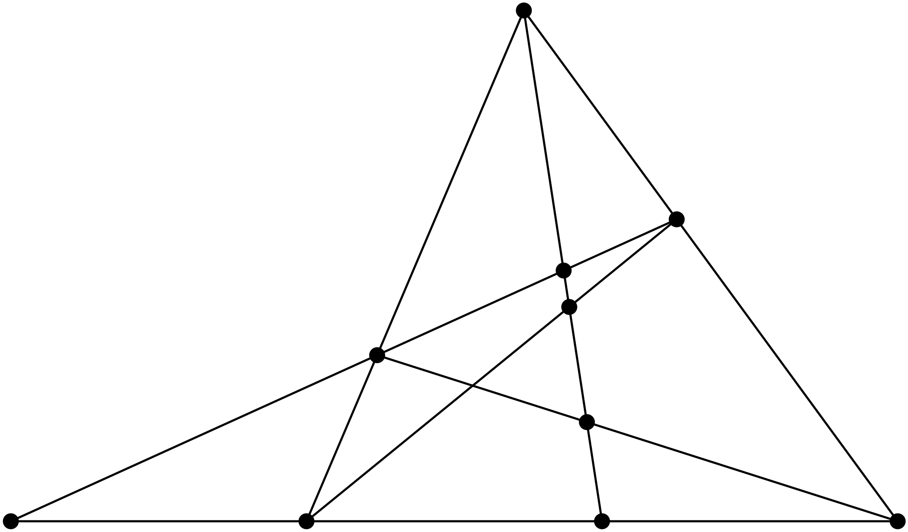


An infinite family - GI

$$GI_r = \left[\begin{array}{ccc|c|c|c|c|c|c} -\alpha & -\alpha & -\alpha & 0\cdots 0 & \alpha\cdots\alpha & 1\cdots 1 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ 1 & \alpha & \alpha^2 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & \alpha\cdots\alpha & 1\cdots 1 & 0\cdots 0 \\ \hline & 0_3^k & & I_k & I_k & I_k^0 & I_k & I_k^0 & D_k \end{array}\right]$$

Note that $k = r - 2$.

GI_3



A Conjecture

Conjecture (Archer, 2005; Welsh, 2010)

Let M be a maximum-sized golden-mean matroid of rank r. If M has rank 3, then M is isomorphic to B_{11}, otherwise M is isomorphic to either GI_r or T_r^2.

Proving the Conjecture

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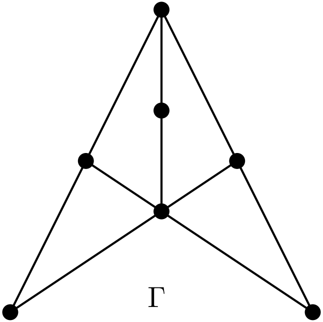
- The existing techniques do work on the T^2 family.

A stepping stone

Theorem (Welsh, 2010)

Let M be a maximum-sized golden-mean matroid of rank r, with no Γ minor. Then M is isomorphic to T_r^2.



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- Computer search.

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- Repeat until all maximum-sized matroids are found.

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- A new approach is needed.

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- Mathematica treatment of finite fields is difficult to work with.

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- Pass this system to Mathematica for solving.