Maximum-Sized Golden-Mean Matroids

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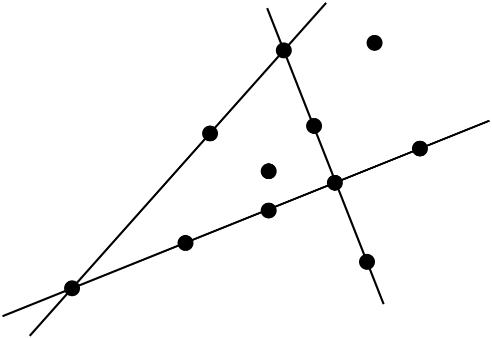
An Introduction to Matroids

Matroids are combinatorial objects that generalise the notion of dependence (found in geometry, linear algebra, graph theory and others).



Let E be a set of finite points in the plane. A set of points is ***dependent*** if any subset of it is contained in a line. Otherwise it is ***independent***.

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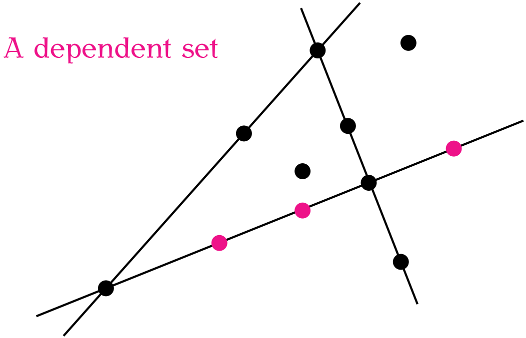

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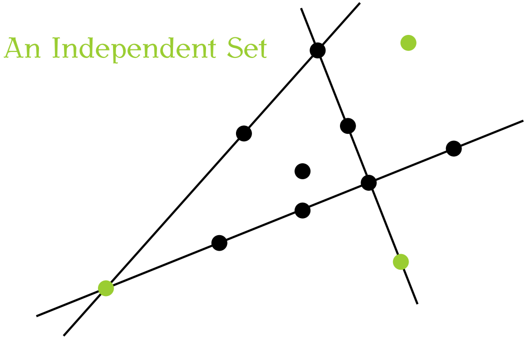

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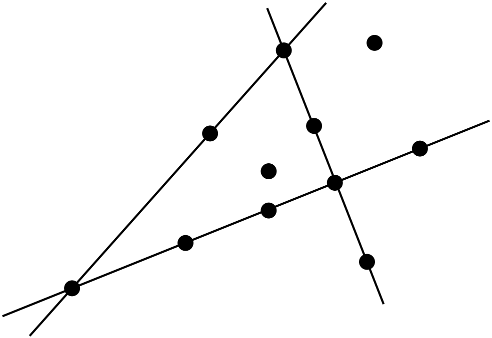

This generalises to higher dimensions easily. In k dimensions, a set of $k + 1$ points is basis if it is not contained in a k-hyperplane.

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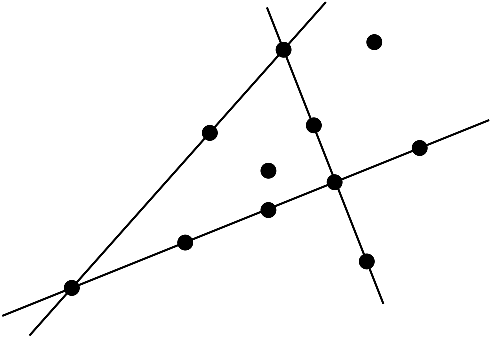

If I_1 and I_2 are independent sets and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e$ is independent.

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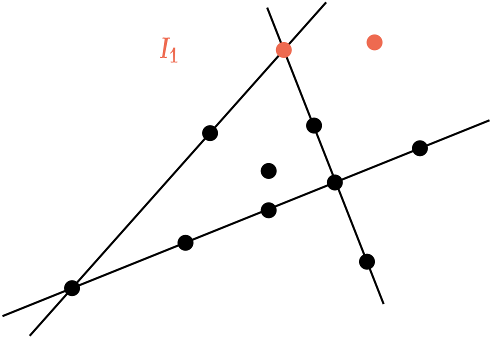

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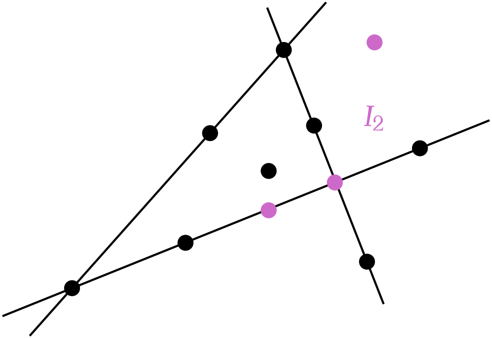

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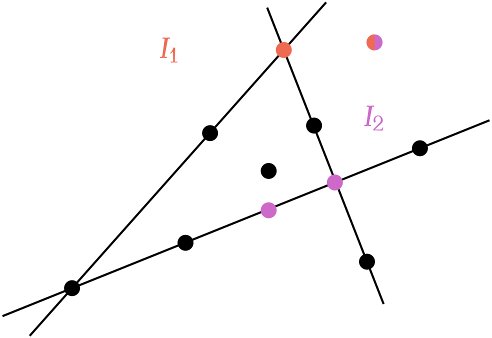

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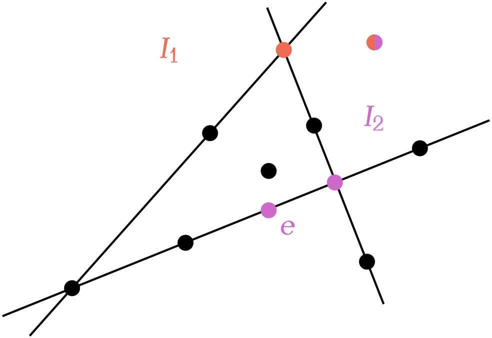

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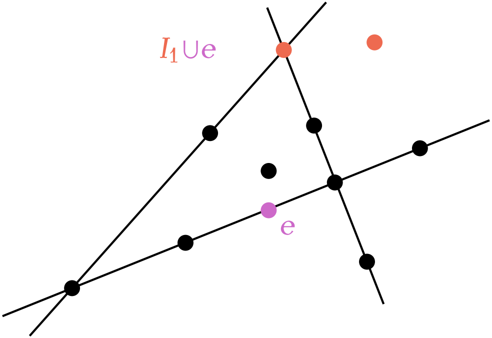

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An Introduction to Matroids — Definition

Definition

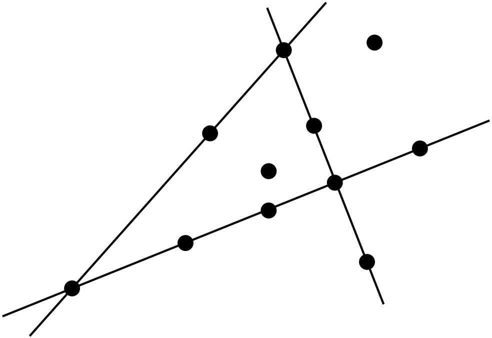
A ***matroid*** $M = (E, \mathcal{I})$ consists of a finite set, E, and a family, $\mathcal{I}$, of subsets of E, satisfying:

- $\varnothing \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The size of the biggest member of $\mathcal{I}$ is called the ***rank*** of M.

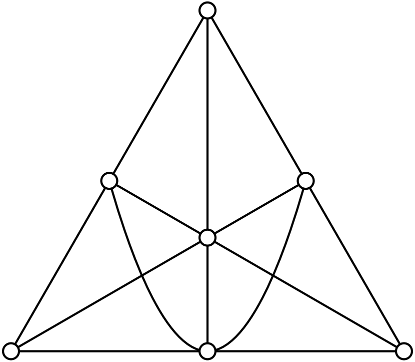
Projective Geometries

Our previous example was points and lines in Euclidean Space.



Projective Geometries

Quite often we consider matroids defined on finite geometries as opposed to Euclidean Space. For example, $GF(2)$.



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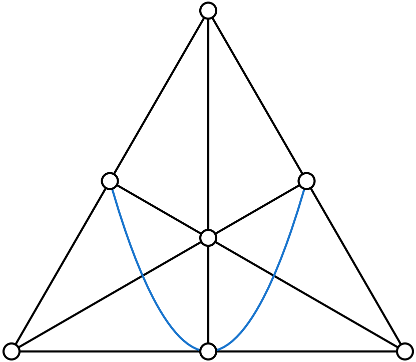

Note that lines need not be straight.

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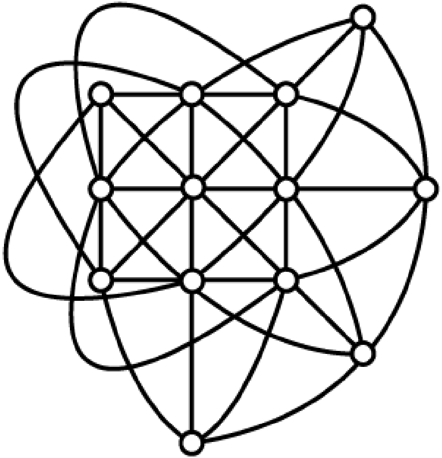

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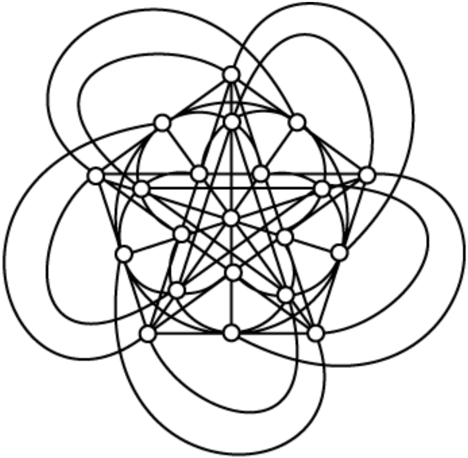

Intersection Classes

Interesting classes of matroids arise from considering the intersections of projective geometries. Some of the most well-studied examples are:

- Matroids that arise from $GF(2) \cap GF(3)$.
- Matroids that arise from $GF(3) \cap GF(5)$.
- Matroids that arise from $GF(3) \cap GF(4) \cap GF(5)$.
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Maximum-Sized Matroids

Problem

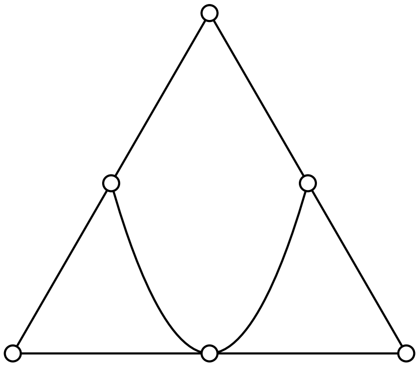
For a given class of matroids, what are the biggest matroids for any given rank?

Solutions to this problem currently exist for various intersection classes of matroids.

Maximum-Sized $GF(2) \cap GF(3)$ Matroids

Theorem (Heller, 1954)

Let M be a maximum-sized matroid of rank r representable over $GF(2)$ and $GF(3)$. Then M is isomorphic to T_r^0.



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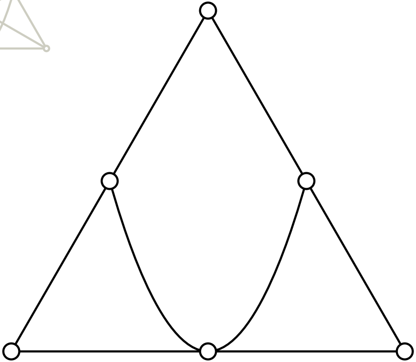

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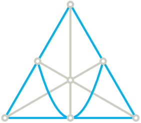

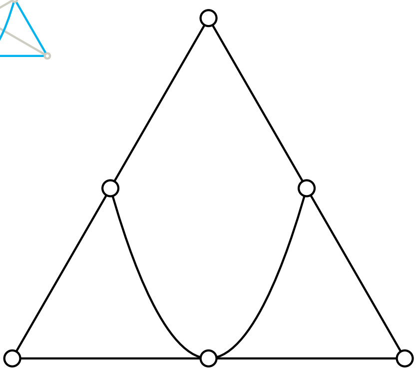

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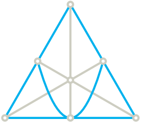

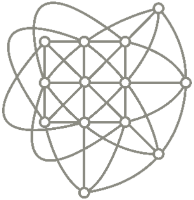

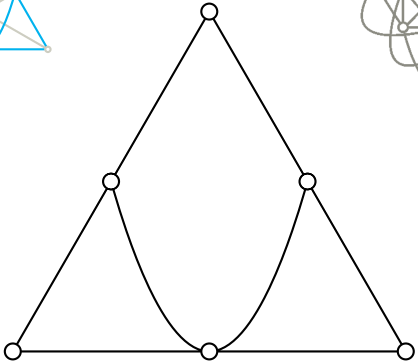

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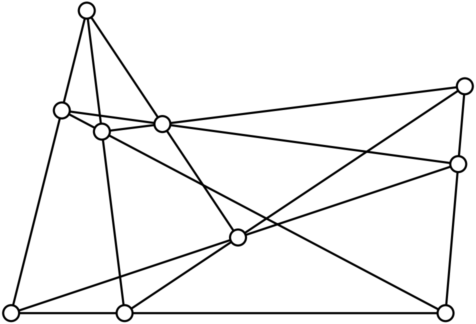
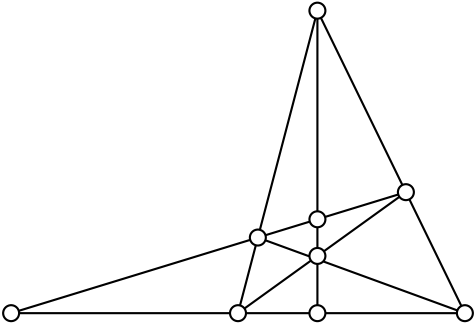


Maximum-Sized $GF(3) \cap GF(5)$ Matroids

Theorem (Kung & Oxley, 1988–1990)

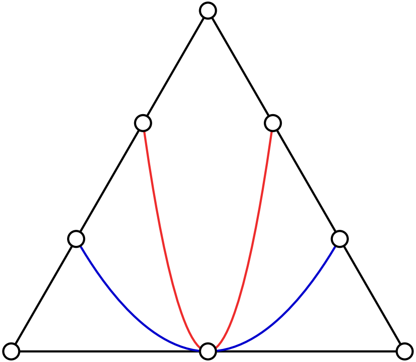
Let M be a maximum-sized matroid of rank r representable over $GF(3) \cap GF(5)$. Then M is isomorphic to $Q_r(GF(3)^*)$.



Maximum-Sized $GF(3) \cap GF(4) \cap GF(5)$ Matroids

Theorem (Oxley, Vertigan, & Whittle, 1998)

Let M be a maximum-sized matroid of rank r representable over $GF(3) \cap GF(4) \cap GF(5)$. Then M is isomorphic to T_r^1.



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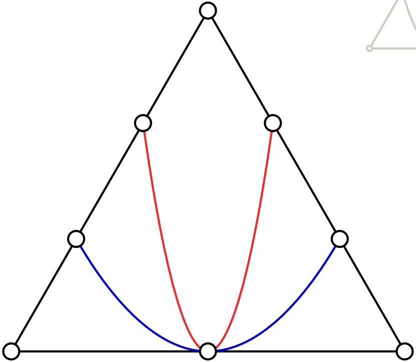

Maximum-Sized $GF(4) \cap GF(5)$ Matroids

$GF(4) \cap GF(5)$ matroids are different to these other classes.

- At rank three, there is a sporadic matroid. Such a matroid does not appear in the other classes.

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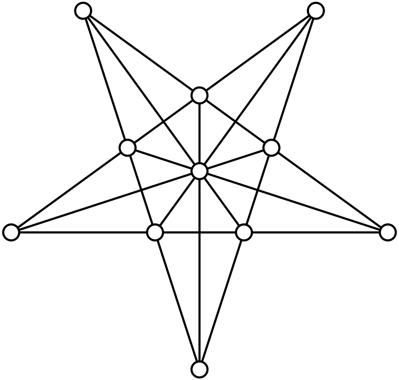
- At rank three, there is a sporadic matroid. Such a matroid does not appear in the other classes.
- At other ranks, there are three families, not the one as in the other classes.

Maximum-Sized $GF(4) \cap GF(5)$ — Betsy Ross



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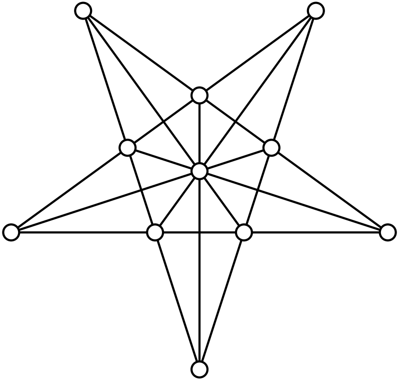
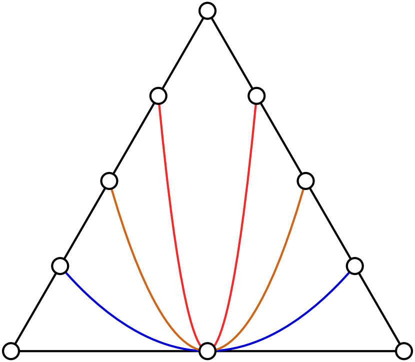


The cross ratios of the four-point lines are powers of the golden ratio. This is a trait in all $GF(4) \cap GF(5)$ matroids, so we shall refer to this class as ***golden-mean***.

Maximum-Sized Golden-Mean — T_r^2



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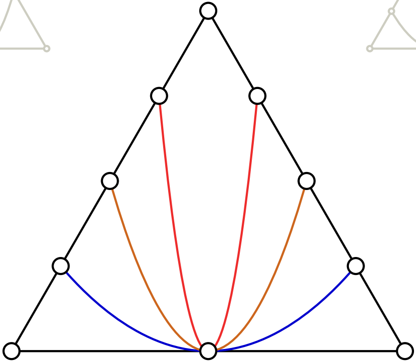
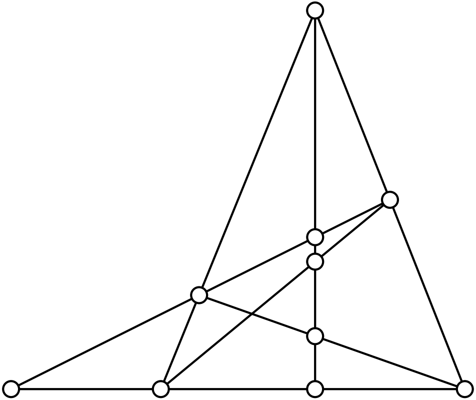
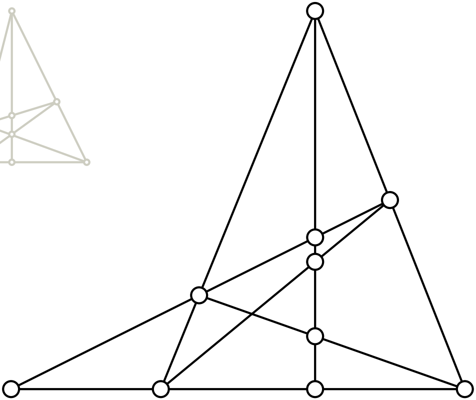
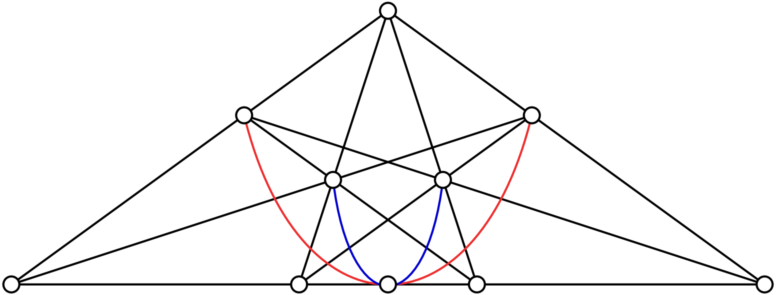




Maximum-Sized Golden-Mean — Dowling



Maximum-Sized Golden-Mean — Hyperpentagon



Conjecture (Archer, 2005)

Let M be a maximum-sized golden-mean matroid. If M has rank three, then M is isomorphic to the Betsy Ross. Otherwise M is isomorphic to either T_r^2, the Dowling of rank r, or the Hyperpentagon of rank r.

Maximum-Sized Golden-Mean Matroids

Theorem (Mayhew & Welsh, 2012)

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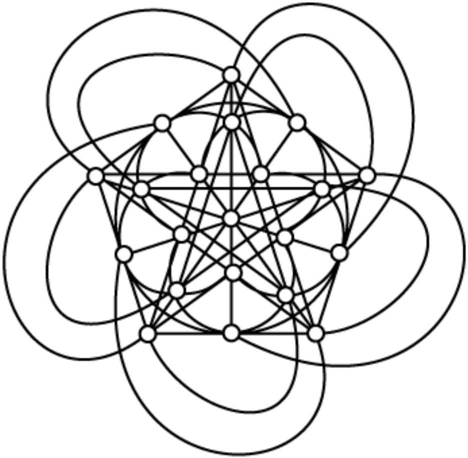
We prove this by induction on r, with the base case (rank 3) being done by a computer search.

Proof Sketch

Recall the picture of the $GF(4)$ projective plane.



This implies that there can be no lines longer than five points in our maximum-sized golden mean matroid.

Proof Sketch

We consider a few cases:

- If there is a point on more than one five-point line.
- If all five-point lines are pairwise disjoint.
- If there are no five-point lines.