Secret – Sharing

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- This is impractical, and the numbers grow exponentially.

Threshold Schemes

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  - knowledge of strictly less than k pieces leaves D completely undetermined (so all possible values are equally likely).
- Such a scheme is called a (k, n) ***threshold scheme***.

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- Given $k-1$ of these D_i values, we can do nothing.

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- Given any k of these D_i values we can generate H and hence D.
- Given $k-1$ of these D_i values we can do nothing as there is a hyperplane passing through all $k-1$ points and any given point on g.

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- The set of all shares is $\mathcal{S}$.

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- Γ is ***monotone*** – $\gamma \in \Gamma \wedge \gamma \subseteq \xi \rightarrow \xi \in \Gamma$.
- A (k, n) threshold scheme has access structure $\Gamma = \{Y \subseteq \wp(P) \mid |Y| \geq k\}$.

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- These matroids have a lot to do with matroid ports, and an excluded minor characterisation was given by Seymour in 1976.

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- A perfect secret sharing scheme is ***ideal*** if:
  - The "length" of K is the same as the "length" of the shares.
- This is deliberately vague, though it can be formalised if one limits the origin of the secret.
- Both threshold schemes introduced earlier are ideal, as all fields elements have the same "length".

Secret Sharing Matrix – Definition

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Let $A = [a_{ij} \mid i \in I, j \in J]$ be a finite matrix with entries from a finite set S such that $|S| > 1$.

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$$n(i, j, X) = \left\{ a_{kj} \mid k \in I, a_{kx} = a_{ix} \text{ for all } x \in X \right\}.$$

Then A is a ***secret sharing matrix*** over S if for all $j \in J$ and all $X \subseteq J - \{j\}$, either $n(i, j, X) = S$ for all $i \in I$, or $|n(i, j, X)| = 1$ for all $i \in I$.

Secret Sharing Matrix – Example

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This is a secret-sharing matrix with $S = \{♡, ♠\}$.

	a	b	c	d	e	f
1	♡	♠	♡	♠	♡	♠
2	♡	♠	♠	♡	♡	♡
3	♡	♡	♡	♡	♠	♠
4	♡	♡	♠	♠	♠	♡
5	♠	♡	♡	♠	♡	♡
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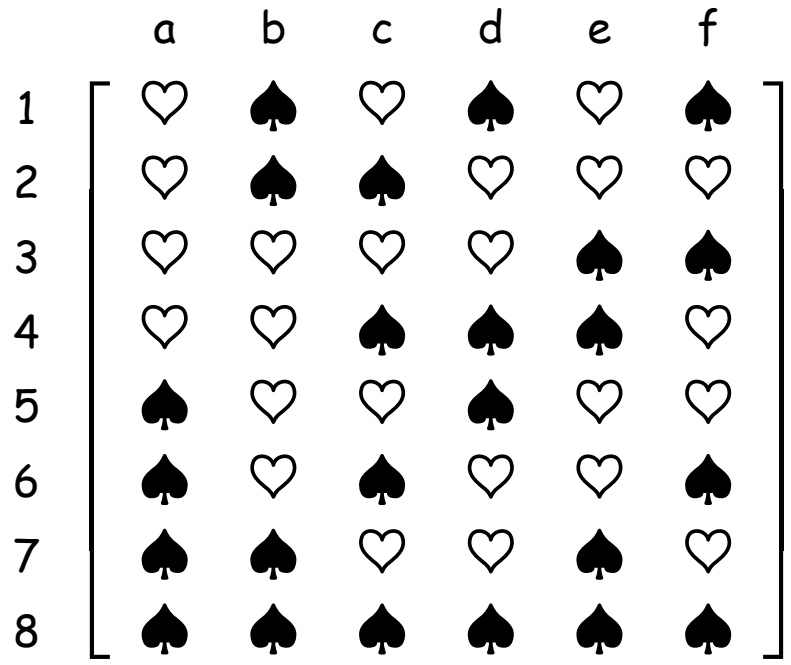

Consider $j = d$ and $X = \{a, b\}$. Then $n(i, d, \{a, b\}) = \{♡, ♠\}$ for all i.

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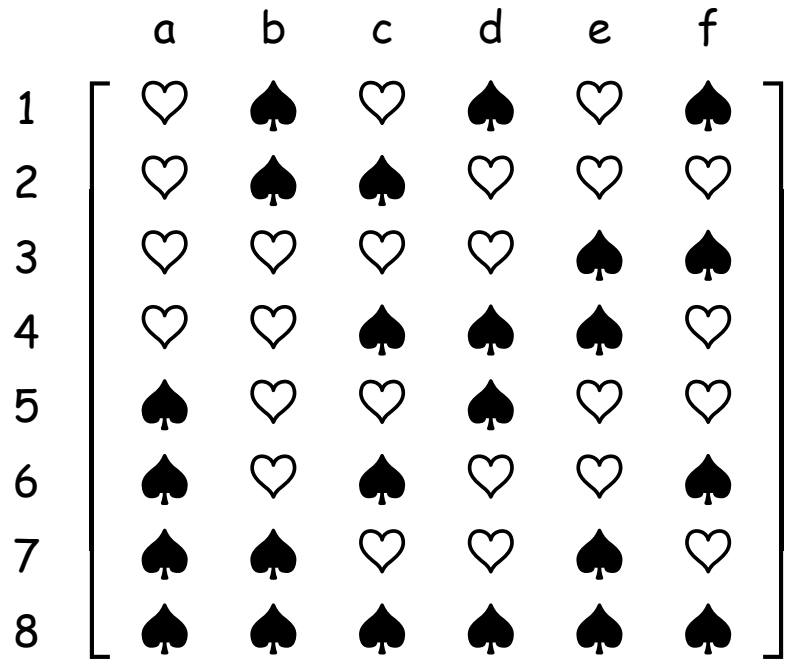
This is a secret-sharing matrix with $S = \{\heartsuit, \spadesuit\}$.



Consider $j = d$ and $X = \{a, b\}$. Then $n(i, d, \{a, b\}) = \{\heartsuit, \spadesuit\}$ for all i.

Consider $j = d$ and $X = \{c, e\}$. Then $|n(i, d, \{c, e\})| = 1$ for all i.

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- We can assume D is the first column.
- D picks row q uniformly at random from A.
- The secret is A_{q1}, and participant p_i gets share A_{qp_i}.
- This gives an ideal secret sharing scheme with $\gamma \in \wp(\mathcal{P})$ being authorised if and only if $|n(i, 1, \gamma)| = 1$ for all i.

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$A =$

	D	p_1	p_2	p_3	p_4	p_5
1	♡	♠	♡	♠	♡	♠
2	♡	♠	♠	♡	♡	♡
3	♡	♡	♡	♡	♠	♠
4	♡	♡	♠	♠	♠	♡
5	♠	♡	♡	♠	♡	♡
6	♠	♡	♠	♡	♡	♠
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- D picks row 3, so the secret is ♡.

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- p_2 and p_4 get together, and can reduce the possible rows to 3 or 7, but the secret is still safe.

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- D picks row 3, so the secret is ♡.
- p_2 and p_4 get together, and can reduce the possible rows to 3 or 7, but the secret is still safe.
- They then torture p_1 and can now determine that the secret is ♡.

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Let $A = [a_{ij} \mid i \in I, j \in J]$ be a secret sharing matrix, and let $\mathcal{I}$ be its collection of independent sets. Then $M = (J, \mathcal{I})$ is a matroid with ground set J and independent sets $\mathcal{I}$.

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- Using the earlier way to define a matroid, this definition applies to all ideal secret sharing schemes.

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This matrix is a secret sharing matrix for $M(K_4)$.

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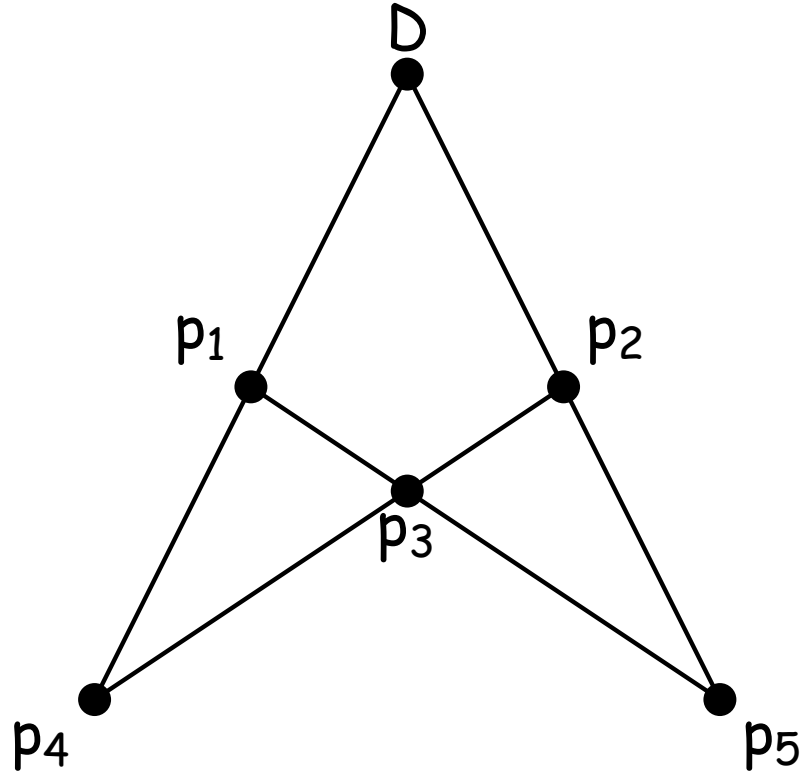

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From this theorem, we can safely disregard non-simple matroids.

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- This also shows that secret sharing matrices are not unique.

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♡	♡	♢	♠	♣

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From this, we can construct a secret sharing matrix.

$U_{2,4}$ part b

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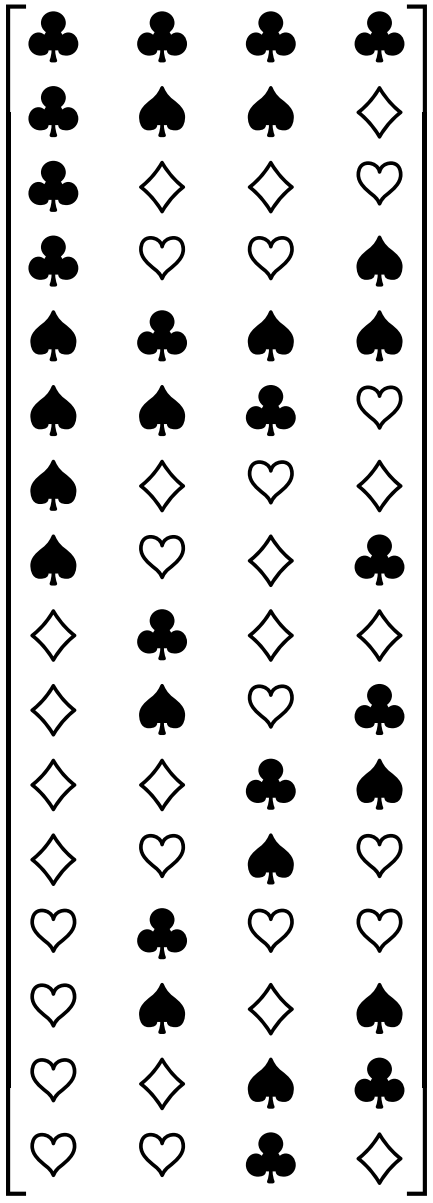

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- When viewed as a matrix, the rowspace is a secret sharing matrix.

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- His proof uses graph theory tricks, and isn't applicable to other matroids.

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Definition

Let $M = (E, r)$ be a matroid with rank function r, and let $d \geq 2$ be an integer. M is ***p-representable*** of degree d if there exists a finite set Ω of cardinality $d^{r(M)}$ and partitions ϖ_i of Ω, $i \in E$, such that for every $F \subseteq E$ the meet-partition $\varpi_F = \bigwedge_{i \in F} \varpi_i$ has $d^{r(F)}$ blocks all of the same cardinality.

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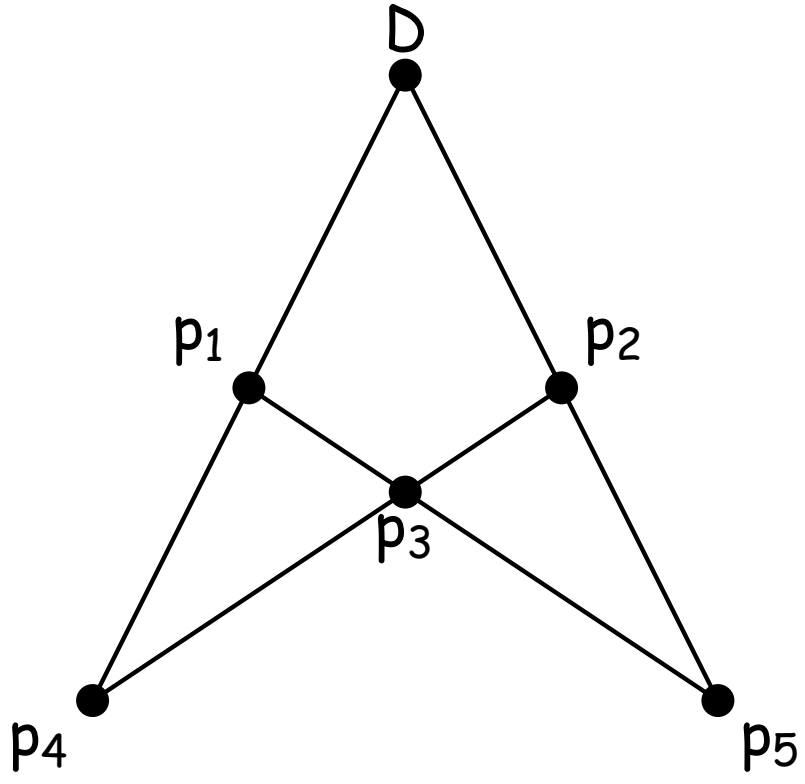
$\varpi_{p_5} = \{\{\alpha, \gamma, \zeta, \vartheta\}, \{\beta, \delta, \varepsilon, \eta\}\}$



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A p-representation of a rank-r matroid M is ***group-induced*** if there is a group G and functions, f_i, $i \in E$, from G^r to G, such that the blocks of ϖ_i are $\{(g_1, \ldots, g_r) \in G^r \mid f_i(g_1, \ldots, g_r) = g\}$, for all $g \in G$, giving a p-representation of M that is equivalent to the original p-representation.

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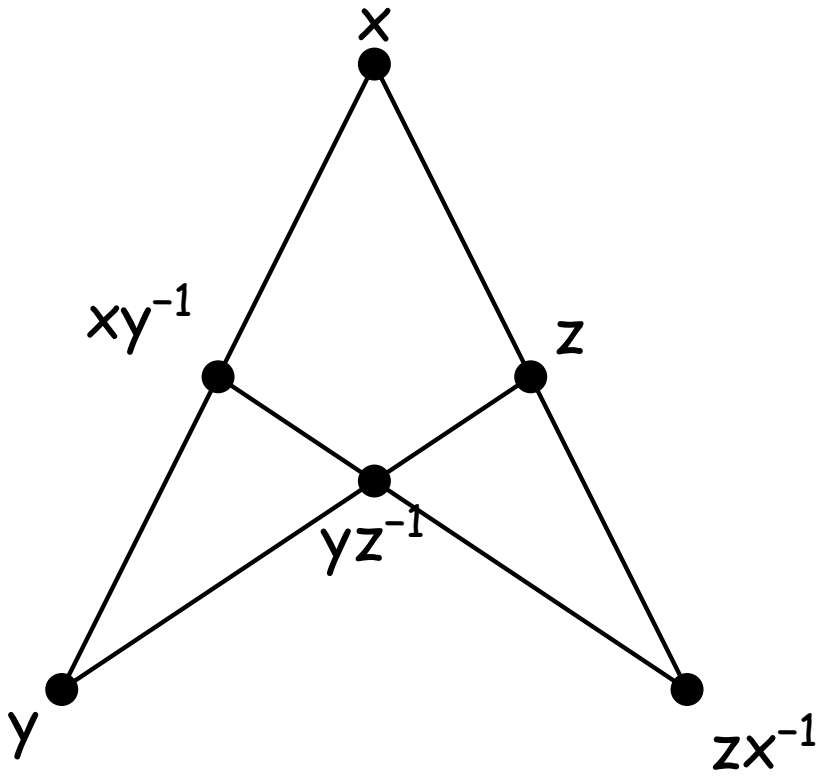

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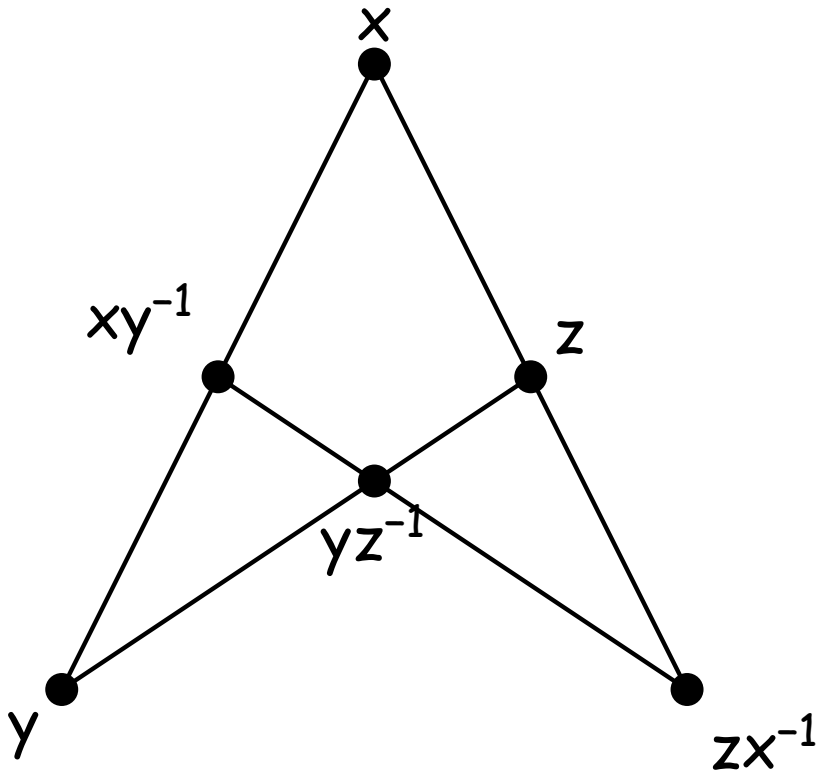

All p-representations of $M(K_4)$ arise from groups, and they are all equivalent to the one given here.

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- A code $\mathcal{C} \subseteq F^S$ is called ***almost affine*** if it satisfies the condition $r(X) \stackrel{\text{def}}{=} \log_q(|\mathcal{C}_X| \in \mathbb{Z}^+ \cup \{0\}$ for all $X \subseteq S$.

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- These codes are known as ***multilinear codes***.

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The matroid of this code is the non-Pappus matroid, showing that representable matroids are a proper subclass of secret sharing matroids.