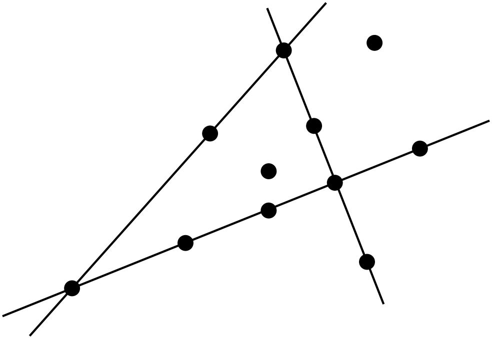
On Maximum-Sized Golden-Mean Matroids

An Introduction to Matroids

Matroids are combinatorial objects that generalise the notion of dependence (found in geometry, linear algebra, graph theory and others).



Let E be a set of finite points in the plane. A set of points is ***dependent*** if any non-trivial dependencies exist between the points. Otherwise it is ***independent***.

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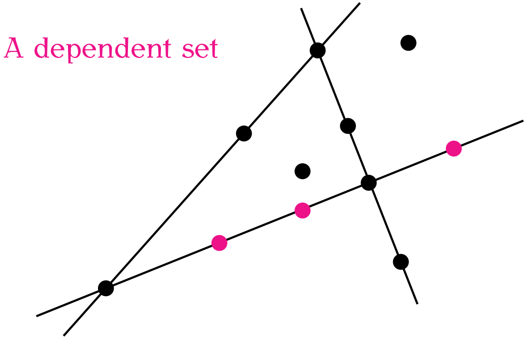

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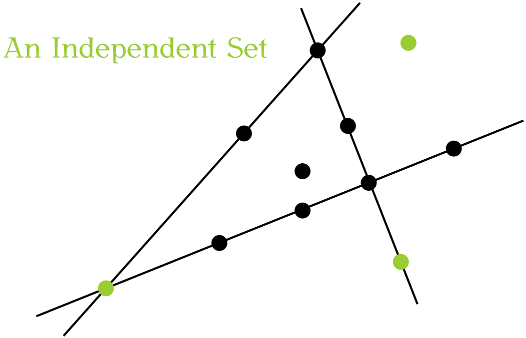

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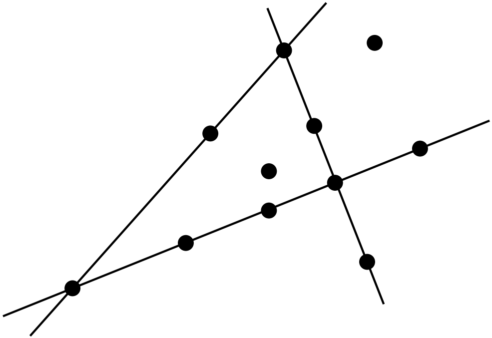

This generalises to higher dimensions easily. In k dimensions, a set of $k+1$ points is basis if it is not contained in a k-hyperplane.

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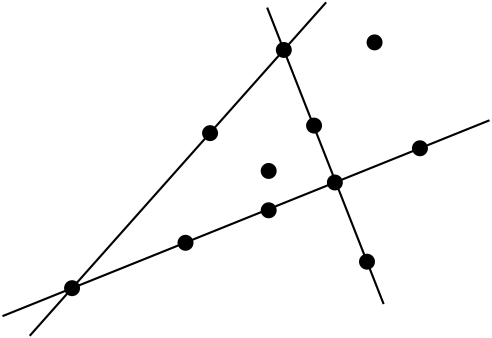

If I_1 and I_2 are independent sets and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e$ is independent.

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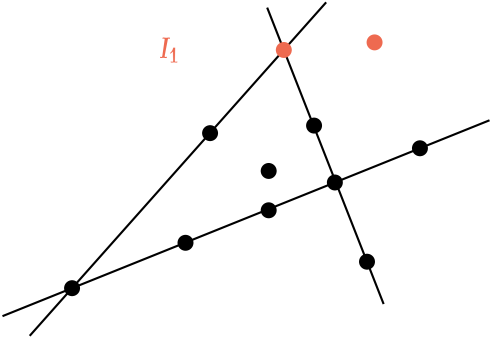

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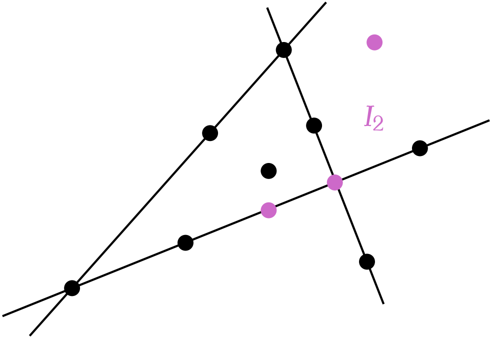

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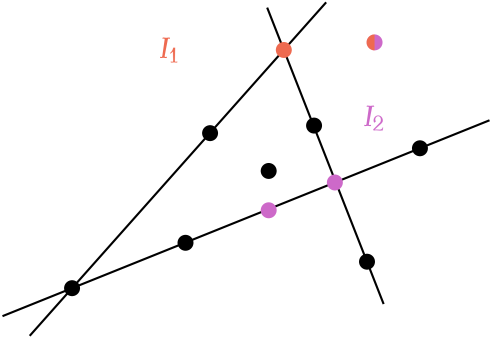

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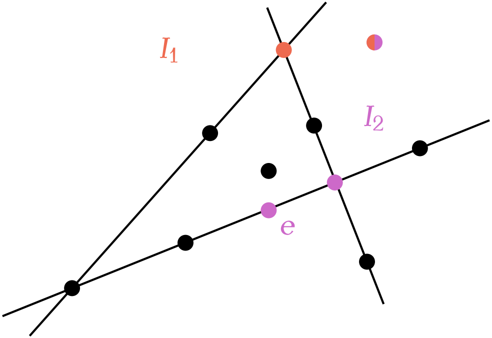

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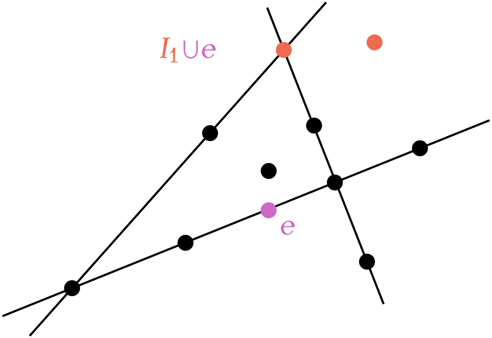

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Definition

A ***matroid*** $M = (E, \mathcal{I})$ consists of a finite set, E, and a family, $\mathcal{I}$, of subsets of E, satisfying:

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The size of the biggest member of $\mathcal{I}$ is called the ***rank*** of M, denoted $r(M)$.

Representable Matroids

- Matroids also generalise linear independence.
- Let A be a matrix over a field $\mathbb{F}$. Then $M(A) = (E, \mathcal{I})$ is a matroid, where E is the set of all columns of A and $\mathcal{I}$ consists of linearly independent subsets of columns of A.

A matrix over $GF(2)$

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$E = \{a, b, c, d\}$
$\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\},$
$\{a, c\}, \{b, c\}\}$

Maximum-Sized

Definition

Let $\mathscr{M}$ be a class of matroids. A rank-r matroid M is said to be ***maximum-sized*** in $\mathscr{M}$ if $\forall N \in \mathscr{M}$ such that $r(N) = r$, then $|M| \geq |N|$.

Definition

Let $\mathscr{M}$ be a class of matroids. The ***growth-rate function*** of $\mathscr{M}$ is the function that defines the number of elements in a maximum-sized matroid of $\mathscr{M}$.

$GF(3)$-representable matroids

Theorem (Whittle, 1997)

Let $\mathcal{F}$ be a set of fields containing $GF(3)$, and let $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. Then for some $q \in \{2, 3, 4, 5, 7, 8\}$, $\mathcal{M}$ is the class of matroids is the class of matroids representable over $GF(3)$ and $GF(q)$.

Historical Results

Maximum-sized results exist for all of Whittle's $GF(3)$-classes of matroids:

- $GF(q)$-representable matroids for each prime power q, in particular $q = 3$.
- Matroids that are representable over $GF(3)$ and $GF(2)$.
- Matroids that are representable over $GF(3)$ and $GF(5)$.
- Matroids that are representable over $GF(3)$ and $GF(4)$.
- Matroids that are representable over $GF(3)$ and $GF(8)$.
- Matroids that are representable over $GF(3)$ and $GF(7)$.

$GF(q)$-representable matroids

Theorem

The matroid $PG(r-1, q)$ is the maximum-sized matroid of rank-r in the class of $GF(q)$-representable matroids. The growth-rate function of the class of $GF(q)$-representable matroids is $\dfrac{q^r - 1}{q - 1}$.

Regular matroids

Definition

A matroid is ***regular*** if it is representable over $GF(3)$ and $GF(2)$.

Theorem (Heller, 1957)

The maximum-sized regular matroid of rank r is $M(K_{r+1})$. The growth-rate function of regular matroids is $\binom{r+1}{2}$.

Dyadic matroids

Definition

A matroid is ***dyadic*** if it is representable over $GF(3)$ and $GF(5)$.

Theorem (Kung and Oxley, 1988-90)

The maximum-sized dyadic matroid of rank r is $Q_r(GF(3)^*)$, the rank-r ternary Dowling geometry. The growth-rate function of dyadic matroids is r^2.

Near-regular and $\sqrt[6]{1}$ matroids

Definition

A matroid is ***sixth-root-of-unity*** if it is representable over $GF(3)$ and $GF(4)$. A matroid is ***near-regular*** if it is representable over $GF(3)$ and $GF(8)$.

Theorem (Oxley, Vertigan, and Whittle; 1998)

Except for rank 3, both the maximum-sized rank-r near-regular matroid and the maximum-sized rank-r $\sqrt[6]{1}$ matroid are T_r^1. Both classes have growth-rate function $\binom{r+2}{2} - 2$.

$GF(3) \cap GF(7)$-matroids

Theorem

The maximum-sized rank-r matroid representable over $GF(3)$ and $GF(7)$ is $Q_r(GF(3)^*)$. This class has growth-rate function r^2.

$\mathbb{P}$-matrices

Definition

Let $\mathbb{P}$ be a subset of a ring that contains -1 and 0. A $\mathbb{P}$-***matrix*** is a matrix with entries from $\mathbb{P}$ such that every subdeterminant is in $\mathbb{P}$.

Example

The subset $\{-1, 0, 1\}$ of $\mathbb{Z}$ defines the regular matroids.

Definition

Let A be a $\mathbb{P}$-matrix. A subset of k columns is independent if it contains a $k \times k$ subdeterminant that is non-zero. This gives rise to a $\mathbb{P}$-***representable*** matroid.

Definition

The ***golden-mean*** set is $\mathbb{G} = \left\{\pm\tau^i \mid i \in \mathbb{Z}\right\} \cup \{0\}$ where τ is the positive root of $x^2 - x - 1$, also known as the golden ratio.

Golden-mean

Theorem (Semple, Vertigan, Pendavingh, Van Zwam)

Let M be a matroid. The following are equivalent:

(i) *M is representable over both $GF(4)$ and $GF(5)$;*

(ii) *M is golden-mean;*

(iii) *M is representable over $GF(p)$ for all primes p such that $p = 5$ or $p \equiv \pm 1 \pmod 5$, and also over $GF(p^2)$ for all primes p.*

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \tau & 1 & 1 & 0 & 0 & \tau & \tau^2 \\ 0 & 0 & 1 & 1 & \tau^2 & 1 & \tau & -\tau & 1 & 1 & \tau^2 \end{bmatrix}$$

- In this matrix, every non-zero subdeterminant is in the set $\{\pm\tau^i \mid i \in \mathbb{Z}\}$.
- For example, $\begin{vmatrix} \tau & \tau \\ \tau^2 & 1 \end{vmatrix} = \tau - \tau^3 = \tau(1-\tau^2) = \tau(-\tau) = -\tau^2$.

Archer's Conjecture

The conjecture we have been trying to prove is due to Archer.

Conjecture

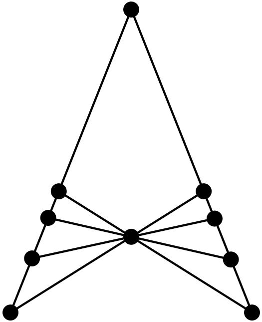
The growth-rate function for the class of golden-mean matroids $\mathcal{G}$ is

$$h_{\mathcal{G}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

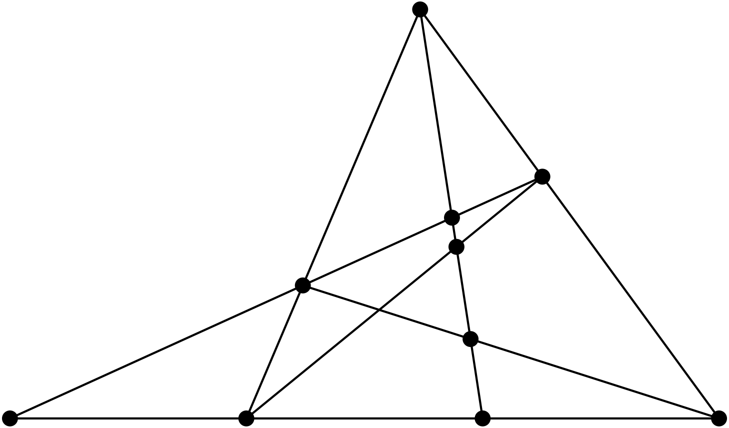
Furthermore, $M \in \mathcal{G}$ is maximum-sized if and only if M is isomorphic to a member of $\mathscr{G}_{r(M)}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

The set $\mathscr{G}_r$ contains three matroids – T_3^2, G_3, and HP_3.

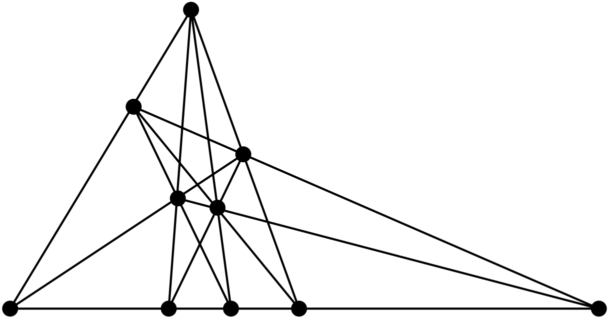
T_3^2



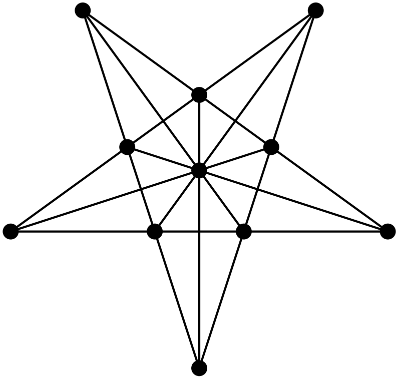
G_3



HP_3



Betsy Ross (B_{11})



The Betsy Ross matroid, or B_{11}.

Removing elements from matroids

- There are two ways to remove an element from a matroid M – ***deletion*** and ***contraction***.
- Deletion is when we simply remove the element from M.
- Contraction is more complex, but can be thought of as projection.

Two Mayhew-Welsh Theorems

Recall that matroids that are representable over $GF(2)$ and $GF(3)$ are regular matroids, while matroids that are representable over $GF(3)$ and $GF(8)$ are near-regular matroids.

Theorem

Let $\mathcal{R}$ be the class of golden-mean matroids with an element e such that, upon contracting e, we obtain a regular matroid. Archer's Conjecture is true in $\mathcal{R}$.

Theorem

Let $\mathcal{N}$ be the class of golden-mean matroids with an element e such that, upon contracting e, we obtain a near-regular matroid. Archer's Conjecture is true in $\mathcal{N}$.

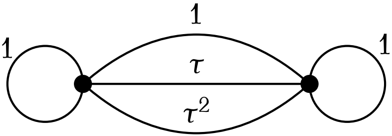
$\mathbb{P}$-graphic

Definition

A matroid M is $\mathbb{P}$***-graphic*** if there exists a $\mathbb{P}$-matrix A with at most two non-zero entries per column such that M is represented over $\mathbb{P}$ by A.

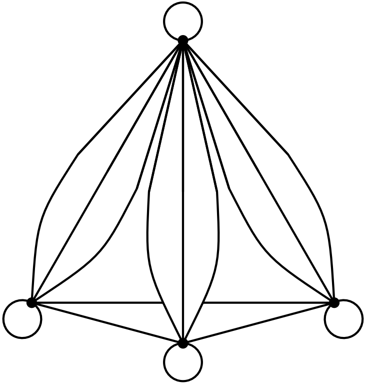
We can scale A so that the first non-zero entry in each column is one.
Note that this naturally gives rise to a (directed) graph with edge labels from $\mathbb{P}$.

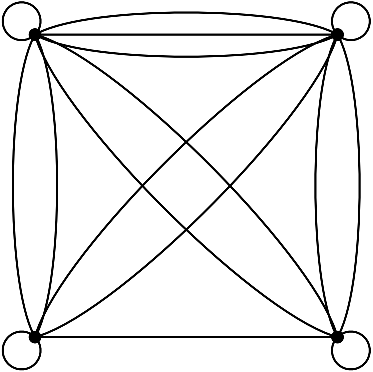
Example



$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \tau & \tau^2 \end{bmatrix}$$

The $\mathbb{G}$-graphic families





These graphs are directed and have edge weights. However, we are only concerned with their structure, so ignore these.

Another Mayhew-Welsh Theorem

Theorem

Let $\mathcal{T}$ be the class of golden-mean graphic matroids. Archer's Conjecture is true in $\mathcal{T}$.

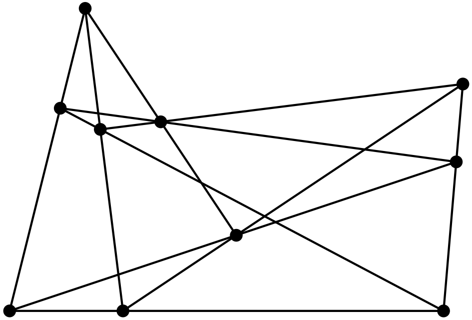
Clique

A ***clique*** is a matroid that can be represented by a matrix of the form

$$[I_k|D_k]$$

where D_k is the $k \times \binom{k}{2}$ matrix whose columns consist of all k-tuples with two non-zero entries, with the first being 1 and the second being -1.

Clique Example



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Final Mayhew-Welsh Theorem

A matroid M has a spanning clique if it is possible to obtain a clique by only deleting elements of M.

Theorem

Let $\mathcal{C}$ be the class of golden-mean matroids with a spanning clique. Archer's Conjecture holds in $\mathcal{C}$.

It is anticipated that this theorem will lead to a proof of Archer's Conjecture for matroids of sufficiently large rank.