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1 Matroids

Matroids are combinatorial structures that generalise the mathematical notion of dependence, found in areas such as geometry, graph theory, and linear algebra.

Definition 1.1 (Whitney [17], Chapter 1 of Oxley [8])**.** A ***matroid*** M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following three properties:

(I1) $\varnothing \in \mathcal{I}$.
(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members of $\mathcal{I}$ are the ***independent sets*** of M, and E is the ***ground set*** of M. A subset of E that is not in $\mathcal{I}$ is called ***dependent***. A minimal dependent set is a ***circuit*** and a maximal independent set is a ***basis***.

Matroids arise from matrices in the following way.

Proposition 1.2 (Oxley [8, Proposition 1.1.1])**.** *Let E be the set of column labels of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labelled by X is a set and is linearly independent in the vector space $V(m, \mathbb{F})$. Then $(E, \mathcal{I})$ is a matroid.*

And from graphs also.

Proposition 1.3. *Let E be the edge set of a graph G, and let $\mathcal{I}$ be the set of all forests of G. Then $(E, \mathcal{I})$ is a matroid, denoted $M(G)$.*

Another important family of matroids are the ***uniform*** matroids.

Proposition 1.4. *Let m and n be non-negative integers with $m \leq n$. Let E be an n-element set and let $\mathcal{I}$ be the set $\{X \subseteq E \mid |X| \leq m\}$. Then $(E, \mathcal{I})$ is a matroid, denoted $U_{m,n}$.*

An element e of a matroid M is a ***loop*** if $\{e\}$ is dependent in M. If f and g are elements of M such that $\{f, g\}$ is dependent in M, then f and g are ***parallel*** in M. A ***parallel class*** of M is a maximal subset X of $E(M)$ such that $|X| \geq 2$ and any two distinct members of X are parallel in M and no member of X is a loop. If M has no loops or parallel classes, then M is ***simple***. If M is a matroid, then $\mathrm{si}(M)$ denotes the simplification of M. The ***rank*** of a subset X of E is the cardinality of the largest independent set contained within X. A simple rank-two subset of E is known as a ***line***, and, if it has at least three elements, a ***long line***. A three-element circuit is known as a ***triangle***.

For matroids of rank three, we can use a geometric representation to illustrate them. This is a two-dimensional drawing of points and lines, with a collection of points being independent if and only if they are not collinear.

Definition 1.5. Let $M = (E, \mathcal{I})$ be a matroid, and let e be a non-loop element of M. Then the ***contraction*** of e from M is the matroid $(E \backslash \{e\}, \{I \subseteq E \backslash \{e\} \mid I \cup \{e\} \in \mathcal{I}\})$.

2 Introduction

Partial fields were introduced by Semple and Whittle [14]. However, we will follow the treatment of Van Zwam [16], starting from a ring.

Definition 2.1 (Van Zwam [16])**.** A ***partial field*** is a pair (R, G), where R is a commutative ring, and G is a subgroup of the group of units of R such that $-1 \in G$.

If $\mathbb{P} = (R, G)$ is a partial field, and $p \in R$, then we say that p is an ***element*** of $\mathbb{P}$, denoted $p \in \mathbb{P}$, if $p = 0$ or $p \in G$. Note that if $p, q \in \mathbb{P}$ then $pq \in \mathbb{P}$, but $p + q$ need not be an element of $\mathbb{P}$.

Example 1. Consider the partial field $\mathbb{U}_0 = (\mathbb{Z}, \{-1, 0, 1\})$, known as the ***regular*** partial field. Then $1 \cdot 1 \in \{-1, 1\}$, but $1 + 1 \notin \{-1, 1\}$. $\Diamond$

Definition 2.2. A matroid M is said to be ***representable over the partial field*** $\mathbb{P}$ if there is a matrix $\mathfrak{M}$ such that all non-zero subdeterminants of $\mathfrak{M}$ are in $\mathbb{P}$ and a labelling of the columns of $\mathfrak{M}$ by $E(M)$ such that any subset $\{x_1, \ldots, x_k\}$ is independent in M if and only if the submatrix $[x_1, \ldots, x_k]$ contains a $k \times k$ subdeterminant that is non-zero in $\mathbb{P}$. We say that $\mathfrak{M}$ is a $\mathbb{P}$-***matrix***, and that M is a $\mathbb{P}$-***matroid***.

We are interested in characterising the maximum-sized matroids for classes of matroids representable over partial fields.

Definition 2.3 (Kung [6])**.** Let $\mathcal{M}$ be a collection of matroids. A member M of $\mathcal{M}$ is ***extremal in*** $\mathcal{M}$ if M is simple and there is no single element simple extension of M that has the same rank as M and is isomorphic to a member of $\mathcal{M}$.

A member M of $\mathcal{M}$ is ***maximum-sized in*** $\mathcal{M}$ if M is simple and every rank-$r(M)$ simple matroid in $\mathcal{M}$ has a groundset that is no larger than the groundset of M.

Characterisations of the maximum-sized matroids representable over various partial fields are already known.

Recall the regular partial field $\mathbb{U}_0$ from Example 1.

Regular matroids capture the property of being representable over both $GF(2)$ and $GF(3)$.

Theorem 2.4 (Tutte [15], Oxley [8, Theorem 6.6.3])**.** *The following statements are equivalent for a matroid M:*

(i) M is regular.
(ii) M is representable over every field.
(iii) M is binary and, for some field $\mathbb{F}$ of characteristic other than two, M is $\mathbb{F}$-representable.

The next theorem follows from work done by Heller [4].

Theorem 2.5. *Let M be a simple rank-r regular matroid. Then*

$$|E(M)| \leq \binom{r+1}{2}.$$

Furthermore, equality in this bound is attained if and only if $M \cong M(K_{r+1})$. $\square$

Definition 2.6 (Kung and Oxley [7], Section 6.10 of Oxley [8])**.** Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of an n-dimensional vector space over $GF(3)$. The ***ternary Dowling matroid*** $Q_n(GF(3)^*)$ is the ternary matroid of rank n represented by the columns $\omega_1, \ldots, \omega_n$ and the columns $\omega_i - \omega_j$ and $\omega_i + \omega_j$, where $i < j$.

The ***dyadic*** partial field is $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$.

Dyadic matroids capture the property of being representable over both $GF(3)$ and $GF(5)$.

Theorem 2.7 (Whittle [18])**.** *The following are equivalent for a matroid M.*

(i) M is dyadic.
(ii) M is representable over $GF(3)$ and $GF(5)$.
(iii) M is representable over $GF(p)$ for all odd primes p.
(iv) M is representable over $GF(3)$ and $\mathbb{Q}$.
(v) M is representable over $GF(3)$ and $\mathbb{R}$.
(vi) M is representable over $GF(3)$ and $GF(q)$ where q is an odd prime power that is congruent to $2 \bmod 3$.

The next theorem follows from work done by Kung [5] and Kung and Oxley [7].

Theorem 2.8. *Let M be a simple rank-r dyadic matroid. Then*

$$|E(M)| \leq r^2.$$

Furthermore, equality in this bound is attained if and only if $M \cong Q_r(GF(3)^*)$. □

The ***near-regular*** partial field is $\mathbb{U}_1 = (\mathbb{Z}[\alpha, \frac{1}{1-\alpha}, \frac{1}{\alpha}], \langle -1, \alpha, 1-\alpha \rangle)$, where α is an indeterminate. Near-regular matroids capture the property of being representable over $GF(3)$, $GF(4)$, and $GF(5)$.

Theorem 2.9 (Whittle [18])**.** *The following are equivalent for a matroid M.*

(i) M is near-regular.
(ii) M is representable over $GF(3)$ and $GF(8)$.
(iii) M is representable over $GF(3)$, $GF(4)$, and $GF(5)$.
(iv) M is representable over $GF(3)$, $GF(4)$, and $\mathbb{Q}$.
(v) M is representable over all fields except possibly $GF(2)$.

The ***sixth-roots-of-unity*** ($\sqrt[6]{1}$) partial field is $\mathbb{S} = (\mathbb{Z}[\zeta], \langle \zeta \rangle)$, where ζ is a root of $x^2 - x + 1$. Sixth-roots-of-unity matroids capture the property of being representable over $GF(3)$ and $GF(4)$.

Theorem 2.10 (Whittle [18])**.** *The following are equivalent for a matroid M.*

(i) M is a $\sqrt[6]{1}$-matroid.
(ii) M is representable over $GF(3)$ and $GF(4)$.
(iii) M is representable over $GF(3)$ and $GF(2^k)$ for some even integer k.

Maximum-sized characterisations for both near-regular and $\sqrt[6]{1}$ matroids were provided by Oxley, Vertigan, and Whittle [9], using the following two results. The matroid T_r^1 is obtained by adding a point freely on a three point line of $M(K_{r+2})$, contracting that point, and simplifying the resulting matroid.

Theorem 2.11. *Let M be a simple rank-r $\sqrt[6]{1}$-matroid. Then*

$$|E(M)| \leq \begin{cases} \binom{r+2}{2} - 2 & \text{if } r \neq 3; \\ 9 & \text{if } r = 3. \end{cases}$$

Moreover, equality is attained in this bound if and only if $M \cong T_r^1$, when $r \neq 3$, or $M \cong AG(2,3)$ when $r = 3$. □

Corollary 2.12. *Let M be a simple rank-r near-regular matroid. Then*

$$|E(M)| \leq \binom{r+2}{2} - 2.$$

Moreover, equality is attained in this bound if and only if $M \cong T_r^1$. □

There are an infinite number of maximum-sized characterisations for classes of matroids, as the maximum-sized rank-r matroid representable over the field $GF(q)$ is the projective geometry $PG(r-1, q)$.

As these results show, the maximum-sized matroids that are representable over all the fields in a subset of $\{GF(2), GF(3), GF(4), GF(5)\}$ have all been characterised, apart from the maximum-sized matroids representable over $GF(4)$ and $GF(5)$, which we now discuss.

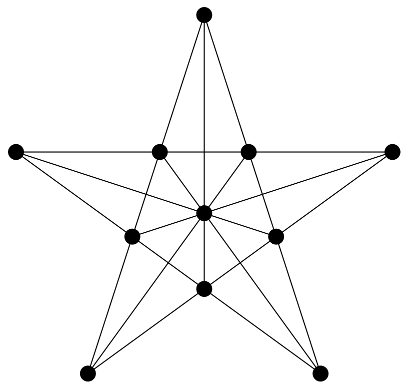

Figure 1: The Betsy Ross

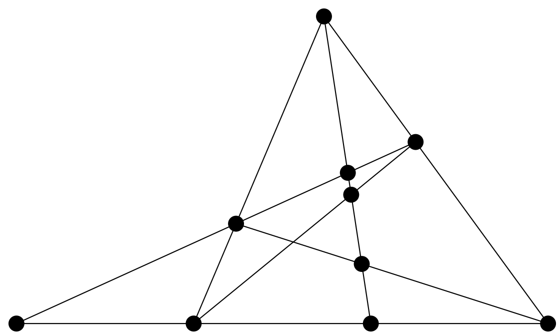

Figure 2: D_3

3 Maximum-sized Golden-mean Matroids

Definition 3.1. The golden-mean partial field, denoted $\mathbb{G}$, is $(\mathbb{Z}[\phi], \langle -1, \phi \rangle)$, where ϕ is the positive root of $x^2 - x - 1$. A matroid is ***golden-mean*** if it has a $\mathbb{G}$-representation.

The following theorem is an unpublished result of Vertigan. In his masters thesis, Semple [11] proved that (ii) implies (iii). For a proof, see Pendavingh and Van Zwam [10, Theorem 1.3].

Theorem 3.2. *Let M be a matroid. The following are equivalent:*

(i) M is representable over both $GF(4)$ and $GF(5)$;
(ii) M is golden-mean;
(iii) M is representable over $GF(p)$ for all primes p such that $p = 5$ or $p \equiv \pm 1 \bmod 5$, and also over $GF(p^2)$ for all primes p. □

The Betsy Ross matroid, or B_{11}, was introduced by Brylawski and Kelly [2]. It was shown by Semple [11] that B_{11} is an extremal rank-three golden-mean matroid. Using computer software, Archer [1] was able to show that B_{11} is the unique maximum-sized rank-three golden-mean matroid.

A geometric representation for B_{11} is given in Figure 1. It has the following $\mathbb{G}$ representation.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \phi & 1 & 1 & 0 & 0 & \phi & \phi^2 \\ 0 & 0 & 1 & 1 & \phi^2 & 1 & \phi & -\phi & 1 & 1 & \phi^2 \end{bmatrix}$$

We now introduce the three infinite families of golden-mean matroids that form the basis of this research. The T_n^2 family was introduced by Semple [12], and the other two were introduced by Archer [1].

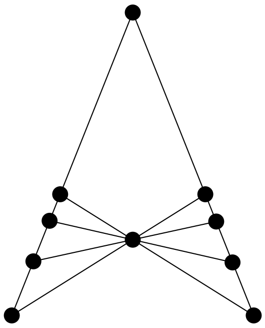

Figure 3: T_3^2

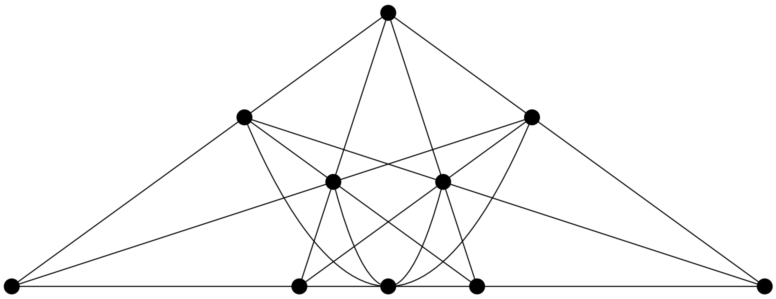

Figure 4: HP_3

Let D_m denote the $m \times \binom{m}{2}$ matrix whose columns consist of all m-tuples with two non-zero entries, with the first being 1 and the second being -1. Let 0_m^n denote the $n \times m$ matrix consisting entirely of zeroes. Let I_m^0 denote the $m \times (m+1)$ matrix $[I_m|\mathbf{0}]$. Let $k = n - 2$.

The first family is the T_n^2 family, a representation of which is given below. A geometric representation of T_3^2 is given in Figure 3.

$$\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \phi & \cdots & \phi & \phi^2 & \cdots & \phi^2 & 0 & \cdots & 0 \\ \hline
0 & & & & & & & & & & & & & & & \\
\vdots & \multicolumn{3}{c|}{I_{n-1}} & \multicolumn{3}{c|}{I_{n-1}} & \multicolumn{3}{c|}{I_{n-1}} & \multicolumn{3}{c|}{I_{n-1}} & \multicolumn{3}{c}{D_{n-1}} \\
0 & & & & & & & & & & & & & & &
\end{array}\right]$$

The second family is the D_n family, a representation of which is given below. A geometric representation of D_3 is given in Figure 2.

$$\left[\begin{array}{ccc|c|c|c|c|c|c}
-\phi & -\phi & -\phi & 0 \cdots 0 & \phi \cdots \phi & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
1 & \phi & \phi^2 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & \phi \cdots \phi & 1 \cdots 1 & 0 \cdots 0 \\ \hline
\multicolumn{3}{c|}{0_3^k} & I_k & I_k & I_k^0 & I_k & I_k^0 & D_k
\end{array}\right]$$

The third family is the HP_n family, a representation of which is given below. A geometric representation of HP_3 is given in Figure 4.

$$\left[\begin{array}{c|c|c|c|c|c}
0 \cdots 0 \; -1 & 1 \cdots 1 & 0 \cdots 0 & \phi \cdots \phi & 1 \; \cdots \; 1 & 0 \cdots 0 \\
0 \cdots 0 \;\; \phi & 0 \cdots 0 & \phi \cdots \phi & \phi \cdots \phi & \phi^2 \cdots \phi^2 & 0 \cdots 0 \\ \hline
I_k^0 & I_k^0 & I_k^0 & I_k^0 & I_k^0 & D_k
\end{array}\right]$$

In his PhD thesis, Archer [1] put forward the following conjecture.

Conjecture 3.3 (Archer, 2005). *Let M be a maximum-sized golden-mean matroid. If $r(M) = 3$ then $M \cong B_{11}$, otherwise M is isomorphic to one of $D_{r(M)}$, $HP_{r(M)}$ or $T^2_{r(M)}$.*

The proof of this conjecture is the aim of this research.

4 Proof Progress

In this section, we list, without proof, all the intermediate steps in the proof of this conjecture that have already been completed.

To prove this conjecture, we are using induction on the rank of M, our maximum-sized golden-mean matroid. The base case is rank three, which is covered by the following Lemma, proven by an exhaustive computer search.

Lemma 4.1. *Let M be a simple golden-mean matroid of rank three. Then M is isomorphic to a restriction of one of the following matroids:*

- *The Betsy Ross,*
- *HP_3,*
- *G_{10},*
- *T^2_3, or*
- *D_3.*

The next task was to characterise the spikes representable over the golden-mean partial field. These are used to construct contradictions in proofs of various lemmas later in the proof.

Definition 4.2 (Ding et al. [3]). For $n \geq 3$, a simple matroid M is an ***n-spike with tip t*** if it satisfies the following properties.

(i) the ground set is the union of n lines, $L_1, \dots, L_n$, all having three points and passing through a common point t;
(ii) for all k in $\{1, \dots, n-1\}$, the union of any k of $L_1, \dots, L_n$ has rank $k+1$; and,
(iii) $r(L_1 \cup \dots \cup L_n) = n$.

We will refer to an n-spike with tip t as an ***n-spike***. Each line of a spike is known as a ***leg***.

Lemma 4.3. *There are no golden-mean 5-spikes.*

Lemma 4.4. *There are only two golden-mean 4-spikes.*

Lemma 4.5. *If M is a golden-mean 4-spike with tip t, $M' - e = M$, where M' is golden-mean, and $\{e, a, b\}$ is a triangle, where $\{a, b\}$ is a leg of M, then $\{t, e\}$ is a circuit.*

Another result required is the collection of all regular or near-regular matroids with no circuits of size greater than four. The graph $K^+_{2,x}$ is the complete bipartite graph $K_{2,x}$, with an extra edge joining the two vertices in the partition of size two. The matroid $K^\sharp_{2,x}$ is obtained by freely adding a point to every three-point line in $M(K^+_{2,x})$.

Lemma 4.6. *If M is a simple connected near-regular matroid with no circuit of size greater than four, then M is either a restriction of T^3_1 or $K^\sharp_{2,x}$, for some x.*

Corollary 4.7. *If M is a connected regular matroid and M has no circuit of size greater than four, then either $\mathrm{si}(M) \cong M(K_4)$, or $\mathrm{si}(M) \cong M(K_{2,b})$ for some b, or $\mathrm{si}(M) \cong M(K^+_{2,b})$ for some b.*

Some connectivity lemmas are also needed.

Lemma 4.8. *Let M be a minimal counterexample to Conjecture 3.3. Then M is 2-connected.*

Lemma 4.9. *Let M be a minimal counterexample to Conjecture 3.3. Then M is 3-connected.*

Lemma 4.10. *Let M be a minimal counterexample to Conjecture 3.3. Then M is vertically 4-connected.*

The proof really starts with these results. It can be broken into a few cases:

- There is some element in M such that M/e is regular.
- M does not fall under the first case, and there is some element in M such that M/e is near-regular.
- M does not fall under one of the first two cases.

The first case has been completed, the second case is partially done, and the third case has a strategy mapped out.

The first case was able to use the existing techniques, such as those used by Oxley, Vertigan, and Whittle [9] and Semple [13]. The next seven results sum up this case.

Definition 4.11. Let M be a matroid, and let e be an element of M, and let $\mathfrak{L}$ be the set of the long lines of M. Let $X = \{e\} \cup \{f \in E(M) \mid \exists L \in \mathfrak{L} \text{ with } e, f \in L\}$. Then $\mathcal{L}(M, e)$ is defined to be the matroid obtained from restricting M to X, contracting e, and then simplifying.

Corollary 4.12. *Let M be a minimal counterexample to Conjecture 3.3. If there exists at least one $e \in E(M)$ such that M/e is regular, then $\mathcal{L}(M, e)$ has no circuits of size greater than four.*

Corollary 4.13. *Let M be a minimal counterexample to Conjecture 3.3. If there exists at least one $e \in E(M)$ such that M/e is regular, then $\mathcal{L}(M, e)$ consists of copies of $M(K_4)$, $M(K_{2,b})$, and $M(K^+_{2,d})$ in addition to a (potentially empty) collection of $U_{1,1}$'s.*

Lemma 4.14. *Let M be a minimal counterexample to Conjecture 3.3. If there exists at least one $e \in E(M)$ such that M/e is regular, then $\mathcal{L}(M, e)$ is a collection of $U_{1,1}$'s.*

Corollary 4.15. *Let M be a minimal counterexample to Conjecture 3.3, and let $e \in E(M)$ be a point such that M/e is regular. Then there are exactly $r - 1$ lines through e, all of length five.*

Lemma 4.16. *Let M be a minimal counterexample to Conjecture 3.3. If there exists at least one $e \in E(M)$ such that M/e is regular, then $\mathrm{si}(M/e) \cong M(K_r)$.*

Lemma 4.17. *Let M be a minimal counterexample to Conjecture 3.3. If there exists at least one $e \in E(M)$ such that M/e is regular, then any two elements in $\mathcal{L}(M, e)$ will be on a triangle in $\mathrm{si}(M/e)$.*

Lemma 4.18. *Let M be a rank-r minimal counterexample to Conjecture 3.3 with a point e such that M/e is regular. Then $M \cong T^2_r$.*

These results show that in the first case, Conjecture 3.3 holds.

Before heading into the next case, we needed some more results about $\mathcal{L}(M, e)$. The existing techniques do not work with this case as readily as they do with the previous case. We slowly reduce the possible components of $\mathcal{L}(M, e)$ until we're left with two configurations that we can then use to prove the rest of the conjecture. The next fifteen results outline this procedure.

Lemma 4.19. *Let M be a minimal counterexample to Conjecture 3.3 such that M/e is not binary. Then there can only be one five-point line going through e.*

Lemma 4.20. *Let M be a minimal counterexample to Conjecture 3.3, and let e be a point of M. Then $\mathcal{L}(M, e)$ is near-regular.*

Corollary 4.21. *Let M be a minimal counterexample to Conjecture 3.3, and let e be a point of M. Then the connected components of $\mathcal{L}(M, e)$ are restrictions of either T^1_3 or $K^\sharp_{2,x}$, for some x.*

P_5 is a rank-three matroid consisting of a three-point line and two points in space.

Lemma 4.22. *$\mathcal{L}(M,e)$ cannot have P_5 as a minor.*

Corollary 4.23. *Let M be a minimal counterexample to Conjecture 3.3, and let e be a point of M. Then the non-regular connected components of $\mathcal{L}(M,e)$ are isomorphic to $U_{2,4}$.*

Lemma 4.24. *The possible components of $\mathcal{L}(M,e)$ are:*

- *$M(K_2,d)$, for some d.*
- *$M(K_4)$.*
- *$M(K^+_{2,b})$, for some b.*
- *$U_{2,4}$.*
- *$U_{1,1}$.*

In a minimal counterexample to Conjecture 3.3, we are required to lose a certain number of points on contraction, by induction. This implies a lower bound on $|\mathcal{L}(M,e)|$. Forbidden configurations in $\mathcal{L}(M,e)$ imply an upper bound. We compare these bounds, using the deficit, which measures the gap between the bounds, and find two solutions.

Lemma 4.25. *The minimum deficit of $M(K_{2,d})$ is 2.*

Lemma 4.26. *The minimum deficit of $M(K_4)$ is 0.*

Lemma 4.27. *The minimum deficit of $M(K^+_{2,b})$ is -1. Furthermore, if the deficit is -1, then there is a five-point line.*

Lemma 4.28. *The minimum deficit of $U_{2,4}$ is 0.*

Lemma 4.29. *The minimum deficit of $U_{1,1}$ is -1.*

The weight of a point is the length of the line going through e and that point in M.

Lemma 4.30. *If one of the components of $\mathcal{L}(M,e)$ is a $M(K^+_{2,b})$ with the "tip" being weighted four or five, then every other point in $\mathcal{L}(M,e)$ is weighted three.*

Lemma 4.31. *If there exists a point in $\mathcal{L}(M,e)$ with weight five, then there can be no $U_{2,4}$ components.*

Lemma 4.32. *If there exists a point in $\mathcal{L}(M,e)$ with weight five, then there can be no $M(K_4)$ components.*

Lemma 4.33. *The two possible configurations of $\mathcal{L}(M,e)$ are:*

- *A collection of $U_{1,1}$'s, one with a weight of five, and the rest with weights of four.*
- *A solitary $M(K^+_{2,r-2})$.*

The second case (M/e is near-regular) can now be broken into two natural sub-cases from Lemma 4.33. The first subcase, where $\mathcal{L}(M,e)$ consists of a collection of $U_{1,1}$'s, has been completed. The next five results outline this process.

Lemma 4.34. *In the case where $\mathcal{L}(M,e)$ is a collection of $U_{1,1}$'s, $\mathcal{L}(M,e)$ forms a basis of M/e.*

Lemma 4.35. *If a basis of T^1_n has a fundamental circuit of size greater than three, then this cannot occur in M/e, inside the case where $\mathcal{L}(M,e)$ is a collection of $U_{1,1}$'s.*

Lemma 4.36. *The only basis of T^1_n with all fundamental circuits having size three is equivalent to the standard basis from [9].*

Lemma 4.37. *The $U_{1,1}$ weighted five is associated with the "tip" of T^1_n.*

Lemma 4.38. *In this case, M is isomorphic to D_r.*

Therein lies the current state of progress towards this conjecture.

5 Future Work

The next target is to consider the second subcase of the near-regular case, namely, that $\mathcal{L}(M, e)$ is a solitary $M(K^{+}_{2,r-2})$. This case appears to lead to both the D_r family and the HP_r family of matroids, depending on how $\mathcal{L}(M, e)$ is embedded into M/e.

Beyond this, there is the final case to consider – when M/e is never near-regular. We intend to use a limited notion of a property called "roundedness" to prove this case, which should be contradictory.

The main difficulties in proving this conjecture, when compared to the other maximum-sized characterisations for partial-field matroids are:

1. The three families (as opposed to only one). This increases the complexity and number of the cases, forcing us to come up with new techniques to tackle the conjecture.
2. Lack of unique-representability. This is not a major problem, as there is a workaround that I am currently exploiting. However, this workaround is not very attractive and leads to a less attractive proof. If there is time remaining after proving this conjecture, it is feasible that I will look into this problem.

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