

On Maximum-Sized Golden-Mean Matroids

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Abstract

A rank- r simple matroid is maximum-sized in a class if it has the largest number of elements out of all simple rank- r matroids in that class. Maximum-sized matroids have been classified for various classes of matroids: regular (Heller, 1957); dyadic (Kung and Oxley, 1988-90); k -regular (Semple, 1998); near-regular and sixth-root-of-unity (Oxley, Vertigan, and Whittle, 1998).

Golden-mean matroids are matroids that are representable over the golden-mean partial field. Equivalently, a golden-mean matroid is a matroid that is representable over $GF(4)$ and $GF(5)$.

Archer conjectured that there are three families of maximum-sized golden-mean matroids. This means that a proof of Archer's conjecture is likely to be significantly more complex than the proofs of existing maximum-sized characterisations, as they all have only one family.

In this thesis, we consider the four following subclasses of golden-mean matroids: those that are lifts of regular matroids, those that are lifts of near-regular matroids, those that are golden-mean-graphic, and those that have a spanning clique. We close each of these classes under minors, and prove that Archer's conjecture holds in each of them.

It is anticipated that the last of our theorems will lead to a proof of Archer's conjecture for golden-mean matroids of sufficiently high rank.

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Lying in bed would be an altogether perfect and supreme experience if only one had a coloured pencil long enough to draw on the ceiling.

G.K. Chesterton

1

Introduction

We start with a definition.

Definition 1.0.1. Let \mathcal{M} be a minor-closed class of matroids and let r be a non-negative integer. The ***growth-rate function*** of \mathcal{M} is

$$h_{\mathcal{M}}(r) = \max \{ \epsilon(M) \mid M \in \mathcal{M} \text{ and } r(M) \leq r \},$$

where $\epsilon(M)$ is the number of rank-one flats of M .

We say that M is ***maximum-sized*** in \mathcal{M} if M is a simple matroid in \mathcal{M} such that $\epsilon(M) = h_{\mathcal{M}}(r(M))$.

The results in this thesis characterise the matroids that maximise the growth-rate function for various classes of golden-mean matroids. For an overview of growth-rate functions, see Section 14.10 in Oxley [10].

1.1 History

It is easy to characterise the maximum-sized matroids in the classes of graphic matroids and matroids representable over a finite field. The next two results are well-known.

Theorem 1.1.1. *Let \mathcal{M} be the class of graphic matroids. Then*

$$h_{\mathcal{M}}(r) = \binom{r+1}{2}.$$

Furthermore, $M \in \mathcal{M}$ is maximum-sized if and only if M is the cycle matroid of a complete graph.

Theorem 1.1.2. *Let \mathcal{M}_q be the class of matroids representable over the finite field $GF(q)$. Then*

$$h_{\mathcal{M}_q}(r) = \frac{q^r - 1}{q - 1}.$$

Furthermore, $M \in \mathcal{M}_q$ is maximum-sized if and only if M is the rank- $r(M)$ projective geometry over $GF(q)$.

Other natural classes of matroids arise when we consider the intersection of two or more finite fields. In order to best understand these classes, we use partial fields.

Partial fields were introduced by Semple and Whittle [17]. However, we will follow the treatment of Pendavingh and Van Zwam [12], starting from a ring. For a more thorough introduction to partial fields, see Pendavingh and Van Zwam [12, 13].

Definition 1.1.3. A **partial field** is a pair (R, G) , where R is a commutative ring with identity, and G is a subgroup of the group of units of R such that $-1 \in G$.

If S is a subset of elements of some group, then $\langle S \rangle$ is the subgroup generated by S . If S is a subset of non-zero elements of a ring, then $\langle S \rangle$ denotes the multiplicative subgroup generated by S . The group of elements with a multiplicative inverse in a ring R is denoted by R^* . If R is a ring and S a set of symbols, then we denote the polynomial ring over R on S by $R[S]$.

Definition 1.1.4. If $\mathbb{P} = (R, G)$ is a partial field, and $p \in R$, then we say that p is an **element of** \mathbb{P} (notation: $p \in \mathbb{P}$) if $p = 0$ or $p \in G$. We define \mathbb{P}^* to be G .

Definition 1.1.5. A matroid M is said to be **representable over the partial field** \mathbb{P} if there is a matrix \mathfrak{M} such that all non-zero subdeterminants of \mathfrak{M} are in \mathbb{P} and a labelling of the columns of \mathfrak{M} by $E(M)$ such that any subset $\{x_1, \dots, x_k\}$ is independent in M if and only if the submatrix $[x_1, \dots, x_k]$ contains a $k \times k$ subdeterminant that is non-zero in \mathbb{P} . We say that \mathfrak{M} is a **\mathbb{P} -matrix**, and that M is a **\mathbb{P} -matroid**.

Note that every field \mathbb{F} can be considered as the partial field $(\mathbb{F}, \mathbb{F}^*)$.

The following result, which shows why we are interested in partial fields, is an unpublished result of Vertigan that appears in Whittle [24]. A stronger result is given by Pendavingh and Van Zwam [13, Corollary 2.20].

Lemma 1.1.6. *If \mathcal{F} is any set of fields, then, for some partial field \mathbb{P} , the class of matroids representable over all fields in \mathcal{F} is the class of \mathbb{P} -representable matroids.*

Maximum-sized matroids have been characterised for various partial fields. We give some examples here. For a catalogue of partial fields, see Pendavingh and Van Zwam [12, Appendix A].

The first partial field we consider is the regular partial field.

Definition 1.1.7. The **regular** partial field is the partial field

$$\mathbb{U}_0 = (\mathbb{Z}, \{-1, 0, 1\}).$$

A \mathbb{U}_0 -matroid is said to be **regular**.

Theorem 1.1.8 (Tutte, in Oxley [10, Theorem 6.6.3]). *The following are equivalent for a matroid M :*

- (i) M is regular.
- (ii) M is representable over every field.
- (iii) M is binary and, for some field \mathbb{F} of characteristic other than two, M is \mathbb{F} -representable.

Theorem 1.1.9 (Heller [6]). *Let \mathcal{U}_0 be the class of matroids representable over \mathbb{U}_0 . Then*

$$h_{\mathcal{U}_0}(r) = \binom{r+1}{2}.$$

Furthermore, $M \in \mathcal{U}_0$ is maximum-sized if and only if $M \cong M(K_{r(M)+1})$.

Next, we consider the near-regular partial field.

Definition 1.1.10. The **near-regular** partial field is the partial field

$$\mathbb{U}_1 = (\mathbb{Z}[\beta, \frac{1}{1-\beta}, \frac{1}{\beta}], \langle -1, \beta, 1 - \beta \rangle),$$

where β is an indeterminate. A \mathbb{U}_1 -matroid is said to be **near-regular**.

Theorem 1.1.11 (Whittle [23, Theorem 1.4]). *The following statements are equivalent for a matroid M :*

- (i) M is near-regular.
- (ii) M is representable over $GF(3)$ and $GF(8)$.
- (iii) M is representable over $GF(3)$, $GF(4)$, and $GF(5)$.
- (iv) M is representable over $GF(3)$, $GF(4)$, and \mathbb{Q} .
- (v) M is representable over all fields except possibly $GF(2)$.

Theorem 1.1.12 (Oxley, Vertigan, and Whittle [11, Corollary 2.2]). *Let \mathcal{U}_1 be the class of matroids representable over \mathbb{U}_1 . Then*

$$h_{\mathcal{U}_1}(r) = \binom{r+2}{2} - 2.$$

Furthermore, $M \in \mathcal{U}_1$ is maximum-sized if and only if $M \cong T_{r(M)}^1$.

The matroid T_r^1 will be defined in Section 4.1.2.

The regular and near-regular partial fields can be generalised to give the k -regular partial field.

Definition 1.1.13. The **k -regular** partial field is the partial field

$$\mathbb{U}_k = (\mathbb{Q}(\alpha_1, \dots, \alpha_k), \langle \{x - y \mid x, y \in \{0, 1, \alpha_1, \dots, \alpha_k\}, x \neq y\} \rangle),$$

where $\alpha_1, \dots, \alpha_k$ are indeterminates. A \mathbb{U}_k -matroid is said to be **k -regular**.

Let D_n denote the $n \times \binom{n}{2}$ matrix whose columns consist of all n -tuples with two non-zero entries, with the first being 1 and the second being -1 . The matroid T_r^k is represented over \mathbb{U}_k by the following matrix.

$$\left[\begin{array}{c|c|c|c|c|c|c} 1 & 0 \cdots 0 & 1 \cdots 1 & \alpha_1 \cdots \alpha_1 & \cdots & \alpha_k \cdots \alpha_k & 0 \cdots 0 \\ \hline 0 & & & & & & \\ \vdots & I_{r-1} & I_{r-1} & I_{r-1} & \cdots & I_{r-1} & D_{r-1} \\ \hline 0 & & & & & & \end{array} \right].$$

Theorem 1.1.14 (Semple [16, Theorem 2.3]). *Let \mathcal{U}_k be the class of matroids representable over \mathbb{U}_k . Then*

$$h_{\mathcal{U}_k}(r) = \binom{r+k+1}{2} - \frac{k(k+3)}{2}.$$

Furthermore, if $r \neq 3$ or $k \neq 2$, then $M \in \mathcal{U}_k$ is maximum-sized if and only if $M \cong T_{r(M)}^k$. If $r = 3$ and $k = 2$, then $M \in \mathcal{U}_k$ is maximum-sized if and only if M is isomorphic to either T_3^2 (Figure 2.2) or S_{10} (Figure 2.18).

Now we consider the sixth-roots-of-unity partial field.

Definition 1.1.15. The **sixth-roots-of-unity** partial field is the partial field

$$\mathbb{S} = (\mathbb{Z}[\zeta], \langle \zeta \rangle),$$

where ζ is a root of $x^2 - x + 1 = 0$. A \mathbb{S} -matroid is said to be a **sixth-roots-of-unity** matroid.

Theorem 1.1.16 (Whittle [23, Theorem 1.2]). *The following statements are equivalent for a matroid M :*

- (i) M is a sixth-roots-of-unity matroid.
- (ii) M is representable over $GF(3)$ and $GF(4)$.
- (iii) M is representable over $GF(3)$ and $GF(2^k)$ for some even integer k .

Theorem 1.1.17 (Oxley, Vertigan, and Whittle [11, Theorem 2.1]). *Let \mathcal{S} be the class of matroids representable over \mathbb{S} . Then*

$$h_{\mathcal{S}}(r) = \begin{cases} \binom{r+2}{2} - 2 & \text{if } r \neq 3; \\ 9 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{S}$ is maximum-sized if and only if $M \cong T_{r(M)}^1$ when $r(M) \neq 3$, or $M \cong AG(2, 3)$ when $r(M) = 3$.

Finally, we consider the dyadic partial field.

Definition 1.1.18. The **dyadic** partial field is the partial field

$$\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle).$$

A \mathbb{D} -matroid is said to be **dyadic**.

Theorem 1.1.19 (Whittle [23, Theorem 1.1]). *The following statements are equivalent for a matroid M :*

- (i) M is dyadic.
- (ii) M is representable over $GF(3)$ and $GF(5)$.
- (iii) M is representable over $GF(p)$ for all odd primes p .
- (iv) M is representable over $GF(3)$ and \mathbb{Q} .
- (v) M is representable over $GF(3)$ and \mathbb{R} .
- (vi) M is representable over $GF(3)$ and $GF(q)$ where q is an odd prime power such that $q \equiv 2 \pmod{3}$.

The matroid $Q_r(GF(3)^*)$ is represented over $GF(3)$ by the matrix $[I_r | H_r]$, where H_r is the ternary matrix whose columns consist of all r -tuples with two non-zero entries, with the first being 1.

The next theorem follows from work done by Kung [8] and Kung and Oxley [9].

Theorem 1.1.20. *Let \mathcal{D} be the class of matroids representable over \mathbb{D} . Then*

$$h_{\mathcal{D}}(r) = r^2.$$

Furthermore, $M \in \mathcal{D}$ is maximum-sized if and only if $M \cong Q_{r(M)}(GF(3)^*)$.

1.2 Golden-mean

Definition 1.2.1. The golden-mean partial field is the partial field

$$\mathbb{G} = (\mathbb{Z}[\tau], \langle -1, \tau \rangle)$$

where τ is the positive root of $x^2 - x - 1$.

A matroid is **golden-mean** if it has a \mathbb{G} -representation.

The following theorem is an unpublished result of Vertigan. In his masters thesis, Semple [15] proved that (ii) implies (iii). For a proof, see Pendavingh and Van Zwam [13, Theorem 1.3].

Theorem 1.2.2. *Let M be a matroid. The following are equivalent:*

- (i) M is representable over both $GF(4)$ and $GF(5)$;
- (ii) M is golden-mean;
- (iii) M is representable over $GF(p)$ for all primes p such that $p = 5$ or $p \equiv \pm 1 \pmod{5}$, and also over $GF(p^2)$ for all primes p .

The following conjecture is the subject of this thesis. It was stated by Archer in his PhD thesis [1].

Conjecture 1.2.3. *Let \mathcal{G} be the class of matroids representable over \mathbb{G} . Then*

$$h_{\mathcal{G}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{G}$ is maximum-sized if and only if M is isomorphic to a member of $\mathcal{G}_{r(M)}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

The set \mathcal{G}_r contains three matroids, T_r^2 , G_r , and HP_r . These three matroids are described in Section 2.2. The Betsy Ross matroid, or BR , will be described in Section 2.3.

In his thesis, Archer proved Conjecture 1.2.3 for ranks three and four using a computer search.

From now on, the function $h(r)$ will be the function $h_{\mathcal{G}}(r)$ from Conjecture 1.2.3. In Table 1.1 we show some values for $h(r)$.

While we do not prove Conjecture 1.2.3 in this thesis, we do prove the conjecture for various subclasses of golden-mean matroids.

In Chapter 2, we introduce some important concepts that will be used throughout the thesis, and in Chapter 3 we prove some connectivity conditions.

r	$h(r)$
2	5
3	11
4	16
5	23
6	31
7	40

Table 1.1: Values of $h(r)$ for $r \in \{2, \dots, 7\}$

The first two subclasses we prove Conjecture 1.2.3 for are \mathcal{R} and \mathcal{N} .

Let \mathcal{M}_1 be the set of golden-mean matroids M with the property that M/e is regular for some $e \in E(M)$. Let \mathcal{R} be the set of all minors of matroids in \mathcal{M}_1 .

Theorem 4.1.1. *Let \mathcal{R} be the class of matroids defined above. Then*

$$h_{\mathcal{R}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{R}$ is maximum-sized if and only if $M \cong T_{r(M)}^2$.

Let \mathcal{M}_2 be the set of golden-mean matroids M with the property that M/e is near-regular for some $e \in E(M)$. Let \mathcal{N} be the set of all minors of matroids in \mathcal{M}_2 .

Theorem 4.1.2. *Let \mathcal{N} be the class of matroids defined above. Then*

$$h_{\mathcal{N}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{N}$ is maximum-sized if and only if M is isomorphic to a member of $\mathcal{G}_{r(M)}$ when $r(M) \neq 3$, or a member of $\mathcal{G}_3 \cup \{BR \setminus p\}$ when $r(M) = 3$.

A geometric representation of $BR \setminus p$ is given in Figure 2.19.

We then look at two more subclasses of golden-mean matroids, and show that Conjecture 1.2.3 is true in each of them. A \mathbb{G} -matroid M is golden-mean-graphic if there exists a \mathbb{G} -representation A of M such that A has at most two non-zero entries in each column. Firstly, in Chapter 5, we prove the following theorem.

Theorem 5.1.2. *Let \mathcal{T} be the class of golden-mean-graphic matroids. Then*

$$h_{\mathcal{T}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{T}$ is maximum-sized if and only if M is isomorphic to either $T_{r(M)}^2$ or $G_{r(M)}$.

We then prove the following theorem in Chapter 6.

Theorem 6.1.4. *Let \mathcal{M} be the set of golden-mean matroids that have a spanning clique. Let \mathcal{C} be the family of minors of matroids in \mathcal{M} . Then*

$$h_{\mathcal{C}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{C}$ is maximum-sized if and only if M is isomorphic to a member of $\mathcal{G}_{r(M)}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

We anticipate that Theorem 6.1.4 will lead to a proof of Conjecture 1.2.3 for golden-mean matroids of sufficiently large rank.

Unless specified otherwise, all work in this thesis is original. Various results in Chapters 3 and 4 are similar to results from Welsh [21] due to the similar nature of the problems covered.

We now define some terminology that will be used throughout this thesis.

Definition 1.2.4. A **point** of a matroid is a rank-one flat. A **line** of a matroid is a rank-two flat. A **long line** of a matroid is a rank-two flat that contains at least three distinct rank-one flats. A **very long line** of a matroid is a rank-two flat that contains at least four distinct rank-one flats. The **length** of a line is the number of distinct rank-one flats contained in the line.

All undefined concepts and notation can be found in *Matroid Theory* by Oxley [10]. We also refer the reader to that book for an introduction to matroid theory.

Projective geometry is all geometry.

Arthur Cayley

2

Preliminaries

2.1 Sage

In this thesis, we prove various results using computers. The software we use is Sage [19], in particular the matroids package [14]. All code is available at http://michael.welsh.co.nz/thesis_code.

2.1.1 Partial Fields

In order to use Sage most conveniently, we need to find a field of prime order that captures \mathbb{G} . To do this, we use Pendavingh and Van Zwam's Lift Theorem. Before stating the theorem, we give some definitions related to partial fields. All these definitions can be found in Van Zwam's PhD thesis [20].

Definition 2.1.1. If \mathfrak{M} is a \mathbb{P} -matrix with columns labelled by elements of X and rows labelled by elements of Y , then we say that \mathfrak{M} is a $X \times Y$ \mathbb{P} -matrix.

Definition 2.1.2. Let \mathfrak{M} be an $X \times Y$ \mathbb{P} -matrix, and let $x \in X$, $y \in Y$ be such that $\mathfrak{M}_{xy} \in \mathbb{P}^*$. Then we define \mathfrak{M}^{xy} to be the $(X - x) \cup y \times (Y - y) \cup x$ matrix with entries

$$(\mathfrak{M}^{xy})_{ij} = \begin{cases} (\mathfrak{M}_{xy})^{-1} & \text{if } ij = yx \\ (\mathfrak{M}_{xy})^{-1} \mathfrak{M}_{xj} & \text{if } i = y, j \neq x \\ -\mathfrak{M}_{iy} (\mathfrak{M}_{xy})^{-1} & \text{if } j = x, i \neq y \\ \mathfrak{M}_{ij} - \mathfrak{M}_{iy} (\mathfrak{M}_{xy})^{-1} \mathfrak{M}_{xj} & \text{otherwise.} \end{cases}$$

We say that \mathfrak{M}^{xy} is obtained from \mathfrak{M} by *pivoting* over xy .

Definition 2.1.3. Let \mathfrak{M} and \mathfrak{M}' be $X \times Y$ \mathbb{P} -matrices. We say that \mathfrak{M} and \mathfrak{M}' are *scaling-equivalent*, denoted by $\mathfrak{M} \sim \mathfrak{M}'$, if \mathfrak{M}' can be obtained from \mathfrak{M} by scaling rows and columns by elements from \mathbb{P}^* .

Definition 2.1.4. Let \mathfrak{M} be an $X \times Y$ \mathbb{P} -matrix. We say that \mathfrak{N} is a *minor* of \mathfrak{M} if \mathfrak{N} can be obtained from \mathfrak{M} by a sequence of the following operations:

- (i) Permuting rows or columns (and permuting labels accordingly);
- (ii) Multiplying the entries of a row or column by an element of \mathbb{P}^* ;
- (iii) Deleting rows or columns;
- (iv) Pivoting over a nonzero entry.

Definition 2.1.5. Let \mathfrak{M} be an $X \times Y$ \mathbb{P} -matrix, and let \mathfrak{M}' be an $X' \times Y'$ \mathbb{P} -matrix. Then \mathfrak{M} and \mathfrak{M}' are *isomorphic* if there exist bijections $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, such that for all $x \in X$ and $y \in Y$, $\mathfrak{M}_{xy} = \mathfrak{M}'_{f(x)g(y)}$.

Definition 2.1.6. Let \mathfrak{N} be isomorphic to a minor of the \mathbb{P} -matrix \mathfrak{M} . We denote this by $\mathfrak{N} \preceq \mathfrak{M}$.

Definition 2.1.7. Let \mathbb{P}_1 and \mathbb{P}_2 be partial fields. A function $\varphi : \mathbb{P}_1 \rightarrow \mathbb{P}_2$ is a *partial-field homomorphism* if

- (i) $\varphi(1) = 1$;
- (ii) For all $p, q \in \mathbb{P}_1$, $\varphi(pq) = \varphi(p)\varphi(q)$;
- (iii) For all $p, q, r \in \mathbb{P}_1$ such that $p + q = r$, $\varphi(p) + \varphi(q) = \varphi(r)$.

Definition 2.1.8. Let \mathbb{P} be a partial field. An element $p \in \mathbb{P}$ is *fundamental* if $1 - p \in \mathbb{P}$. We denote the set of fundamental elements of a partial field by $\mathcal{F}(\mathbb{P})$.

Definition 2.1.9. Let \mathfrak{M} be a \mathbb{P} -matrix. We define the *cross ratios* of \mathfrak{M} as the set

$$\text{Cr}(\mathfrak{M}) = \left\{ p \mid \begin{bmatrix} 1 & 1 \\ p & 1 \end{bmatrix} \preceq \mathfrak{M} \right\}.$$

2.1.2 The Lift Theorem

We can now state the Lift Theorem.

Theorem 2.1.10 (Lift Theorem [13, Theorem 3.5]). *Let \mathbb{P} and $\widehat{\mathbb{P}}$ be two partial fields, let \mathfrak{M} be a \mathbb{P} -matrix, and let $\varphi : \widehat{\mathbb{P}} \rightarrow \mathbb{P}$ be a homomorphism such that the restriction of φ to the fundamental elements, $\varphi|_{\mathcal{F}(\widehat{\mathbb{P}})} : \mathcal{F}(\widehat{\mathbb{P}}) \rightarrow \mathcal{F}(\mathbb{P})$, is a bijection. Then exactly one of the following is true:*

- (i) *There is a $\widehat{\mathbb{P}}$ -matrix $\widehat{\mathfrak{M}}$ such that $\varphi(\widehat{\mathfrak{M}}) \sim \mathfrak{M}$.*
- (ii) *\mathfrak{M} has a minor \mathfrak{N} such that*
 - a) *There is no $\widehat{\mathbb{P}}$ -matrix $\widehat{\mathfrak{N}}$ such that $\varphi(\widehat{\mathfrak{N}}) = \mathfrak{N}$;*
 - b) *\mathfrak{N} or \mathfrak{N}^T equals*

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q \end{bmatrix}$$

for some distinct $p, q \in \mathcal{F}(\mathbb{P}) - \{0, 1\}$.

The following result, that captures \mathbb{G} inside of $GF(19)$, is due to Pendavingh (personal communication).

Lemma 2.1.11. *Let M be a golden-mean matroid. Then M can be represented by a matrix \mathfrak{M} over $GF(19)$ such that all cross ratios of \mathfrak{M} are in $S = \{0, 1, 4, 5, 6, 14, 15, 16\}$. Furthermore, all matrices of this type represent golden-mean matroids.*

Proof. There is a unique partial field homomorphism from \mathbb{G} to $GF(19)$ that sends τ to 5. Let ψ be this homomorphism. The fundamental elements of \mathbb{G} are $0, 1, \tau, -\tau, \tau^{-1}, -\tau^{-1}, \tau^2$, and τ^{-2} ([20, Lemma 2.5.19]). Calculation shows that ψ operates on $\mathcal{F}(\mathbb{G})$ as shown in Table 2.1.

Hence ψ satisfies the conditions of Theorem 2.1.10 with $\widehat{\mathbb{P}}$ equal to \mathbb{G} and \mathbb{P} equal to $GF(19)$.

x	$\psi(x)$	x	$\psi(x)$	x	$\psi(x)$	x	$\psi(x)$
0	0	τ	5	τ^{-1}	4	τ^2	6
1	1	$-\tau$	14	$-\tau^{-1}$	15	τ^{-2}	16

Table 2.1: Values of ψ , the unique homomorphism from \mathbb{G} to $GF(19)$ such that $\psi(\tau) = 5$

Let \mathfrak{M} be a \mathbb{G} -matrix that represents the \mathbb{G} -matroid M . Then $\psi(\mathfrak{M})$ represents M over $GF(19)$. Since every cross-ratio of \mathfrak{M} is a fundamental element of \mathbb{G} , it follows that every cross-ratio of $\psi(\mathfrak{M})$ is the image of a fundamental element under the homomorphism ψ , so one direction of the lemma is proved.

For the converse, we let \mathfrak{M} be a $GF(19)$ matrix whose cross-ratios are all in S , and let M be the matroid represented by \mathfrak{M} . If condition (i) holds in \mathfrak{M} , then M can be represented by a \mathbb{G} -matrix, so there is nothing left to prove. Therefore we assume that condition (ii) holds. The Sage code `Matroid(reduced_matrix = matrix(GF(19), [[0, 1, 1, 1], [1, 0, 1, 1], [1, 1, 0, 1]]).cross_ratios()` tells us that the cross-ratios of the Fano-type matrix are $\{2, 8, 10\}$, so \mathfrak{M} has a minor equal to

$$\mathfrak{N} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q \end{bmatrix}$$

for some distinct p and q . Since p and q are both cross-ratios of \mathfrak{N} , it follows that $\{p, q\} \subset \{4, 5, 6, 14, 15, 16\}$. We consider all possible values of p and q such that the cross-ratios of \mathfrak{N} are in S , and we show that in each case, if we replace p and q with their pre-images from Table 2.1, then we obtain a \mathbb{G} -matrix. To do this, we compute $z = x - y$ for $\{x, y\} \subset \{\tau, -\tau, \tau^{-1}, -\tau^{-1}, \tau^2, \tau^{-2}\}$ and check if z is in $\{\pm\tau^i \mid i \in \mathbb{Z}\}$. We do this using the Sage code in Appendix 2.A.3. This shows that condition (ii) in Theorem 2.1.10 does not hold. \square

Using Lemma 2.1.11, we are able to check if a given $GF(19)$ matrix represents a golden-mean matroid or not. To do this, we use the function `is_gm()`, detailed in Appendix 2.A.2.

2.1.3 $GF(4)$

Throughout this work, $GF(4)$ will consist of $\{0, 1, \alpha, \alpha^2\}$, where $\alpha^2 = \alpha + 1$. A common problem in representable matroid research is that of inequivalent representations. However, matroids are uniquely representable over $GF(4)$.

Lemma 2.1.12 (Kahn [7, Theorem 1]). *If M is 3-connected and representable over $GF(4)$ then it is uniquely representable over $GF(4)$.*

Because of Lemma 2.1.12, we would like to do all matrix work over $GF(4)$. As such, we use the following homomorphism from \mathbb{G} to $GF(4)$ to create $GF(4)$ -matrices from \mathbb{G} -matrices.

$$\begin{aligned} f : 0 &\mapsto 0 \\ f : 1 &\mapsto 1 \\ f : \tau^k &\mapsto \alpha^k. \end{aligned}$$

Verifying that this is a homomorphism is an easy exercise.

2.2 The Three Families

In this thesis, there are three families of interest. In this section, we introduce the families and give various representations of them.

The three families are the T_r^2 , G_r , and HP_r families. The T_r^2 family was introduced by Semple [16], and the G_r and HP_r families were introduced by Archer [1]. All three families were shown to be golden-mean by Archer [1, Proposition 5.3].

To make the notation easier, we use \mathcal{G}_r to refer to the set $\{T_r^2, G_r, HP_r\}$.

2.2.1 \mathbb{G} -matrices

We introduce the three families as \mathbb{G} -matrices. These are shown in Figure 2.1. The block D_k represents the $k \times \binom{k}{2}$ matrix whose columns consist of all k -

tuples with two non-zero entries, with the first being 1 and the second being -1 . The block 0_m^n denotes the $n \times m$ matrix consisting entirely of zeros.

The first family is the T_r^2 family. Note that the matroid T_r^2 contains $r - 1$ copunctual five-point lines. A geometric representation of T_3^2 is given in Figure 2.2.

The second family is the G_r family. Note that the matroid G_r has one five-point line and $2(r - 2)$ four-point lines, all of which meet the five-point line. A geometric representation of G_3 is given in Figure 2.3.

The final family is the HP_r family. Note that the matroid HP_r has one five-point line and no four-point lines. A geometric representation of HP_3 is given in Figure 2.4.

2.2.2 $GF(4)$ Matrices

We apply the homomorphism f from Section 2.1.3 to the \mathbb{G} -matrix of each of the three families. This yields the $GF(4)$ matrices in Figure 2.5.

2.2.3 Minors

We prove the following lemma.

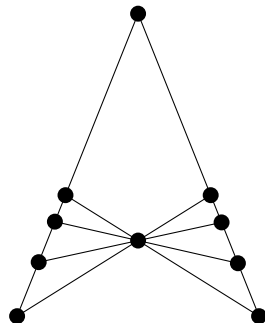
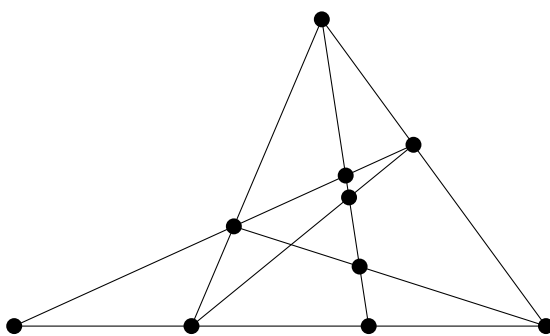
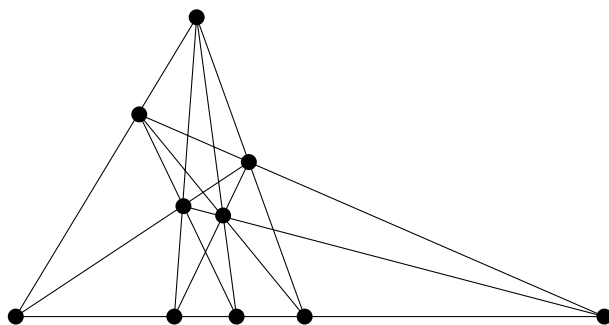
Lemma 2.2.1. *Let M be a member of \mathcal{G}_r . Let M' be isomorphic to $\text{si}(M/e)$ for some $e \in E(M)$.*

- (i) *If M is isomorphic to T_r^2 , then M' is isomorphic to T_{r-1}^2 or $M(K_r)$.*
- (ii) *If M is isomorphic to G_r , then M' is isomorphic to G_{r-1} , T_{r-1}^2 , or T_{r-1}^1 .*
- (iii) *If M is isomorphic to HP_r , then M' is isomorphic to HP_{r-1} , T_{r-1}^2 , or T_{r-1}^1 .*

Proof. We prove this by considering the $GF(4)$ matrices from Figure 2.5 and pivoting on all possible entries. We do this in three near-identical sublemmas.

$$\begin{aligned}
& \left[\begin{array}{c|cccc|c|cccc|c|cccc|c} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \tau & \cdots & \tau & \tau^2 & \cdots & \tau^2 & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & & & & \end{array} \right] \\
& \qquad \qquad \qquad 2.1.1: T_r^2 \\
& \left[\begin{array}{c|cccc|c|cccc|c|cccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & \tau & \cdots & \tau \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \tau & \cdots & \tau & 0 & 0 & \cdots & 0 & 1 & \tau^2 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & & & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & & & & & & & & \end{array} \right] \\
& \qquad \qquad \qquad 2.1.2: G_r \\
& \left[\begin{array}{c|cccc|c|cccc|c|cccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \tau & \cdots & \tau & 0 & \cdots & 0 & 1 & 1 & 1 & \tau & \cdots & \tau^2 \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \tau & \cdots & \tau & 0 & \cdots & 0 & 1 & \tau & \tau^2 & 0 & \cdots & 0 & 1 \\ \hline 0 & & & & & & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & & & & & & & & \end{array} \right] \\
& \qquad \qquad \qquad 2.1.3: HP_r
\end{aligned}$$

Figure 2.1: \mathbb{G} -matrices of the golden-mean families

Figure 2.2: T_3^2 Figure 2.3: G_3 Figure 2.4: HP_3

$$\begin{aligned}
& \left[\begin{array}{c|cccc|c|c|c|c|c} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & \alpha^2 & \cdots & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & & & \end{array} \right] \\
& \qquad \qquad \qquad 2.5.1: T_r^2 \\
& \left[\begin{array}{c|cccc|c|c|c|c|c} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & \alpha & \cdots & \alpha \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & & & & & & & & & & & & & & & \\ \vdots & \vdots & & & & & & & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & & & & & & & & \end{array} \right] \\
& \qquad \qquad \qquad 2.5.2: G_r \\
& \left[\begin{array}{c|cccc|c|c|c|c|c} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha & \cdots & \alpha & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & \alpha^2 & \cdots & \alpha^2 \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha^2 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{array} \right] \\
& \qquad \qquad \qquad 2.5.3: HP_r
\end{aligned}$$

Figure 2.5: $GF(4)$ matrices of the golden-mean families

Throughout this proof, let

$$I'_k = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & I_{k-1} & & 0 \\ \hline & & & 1 \\ 0 & \cdots & 0 & 1 \end{array} \right].$$

Sublemma 2.2.1.1. *If M is isomorphic to T_r^2 , then M' is isomorphic to T_{r-1}^2 or $M(K_r)$.*

Subproof. Consider the $GF(4)$ matrix \mathfrak{M} for T_r^2 in Figure 2.5.1. Let A be the set consisting of the first column of \mathfrak{M} , and let B, C, D, E and F be the subsequent blocks of columns of \mathfrak{M} . Note that $|A| = 1$, $|B| = |C| = |D| = |E| = r - 1$, and $|F| = \binom{r-1}{2}$.

We now contract an element from each set. It is relatively easy to see that if X is a block, and $x, y \in X$ are elements of M , then $\text{si}(M/x) \cong \text{si}(M/y)$.

Let e be an element of $E(M)$.

$e \in A$ In this case, we can see that $\text{si}(M/e) \cong M(K_r)$.

$e \in B$ Let e be the last element of B . We can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in C$ Let e be the last element of C . We pivot by adding the last row to the first row, getting the matrix shown in Figure 2.6.1. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in D$ Let e be the last element of D . We pivot by adding the last row scaled by α to the first row, getting the matrix shown in Figure 2.6.2. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in E$ Let e be the last element of E . We pivot by adding the last row scaled by α^2 to the first row, getting the matrix shown in Figure 2.6.3. Now,

$$\begin{aligned}
& \left[\begin{array}{c|cccccccc} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & \alpha^2 & \cdots & \alpha^2 & \alpha & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline 0 & & & & & & & & & & & & & & & & & & \\ \vdots & & & & I_{r-1} & & & & & I_{r-1} & & & & & D_{r-2} & & & I_{r-2} \\ \hline 0 & & & & & & & & & & & & & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right] \\
& \qquad \qquad \qquad 2.6.1: e \in C \\
& \left[\begin{array}{c|cccccccc} 1 & 0 & \cdots & 0 & \alpha & 1 & \cdots & 1 & \alpha^2 & \alpha & \cdots & \alpha^2 & 1 & 0 & \cdots & 0 & \alpha & \cdots & \alpha \\ \hline 0 & & & & & & & & & & & & & & & & & & \\ \vdots & & & & I_{r-1} & & & & & I_{r-1} & & & & & D_{r-2} & & & I_{r-2} \\ \hline 0 & & & & & & & & & & & & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right] \\
& \qquad \qquad \qquad 2.6.2: e \in D \\
& \left[\begin{array}{c|cccccccc} 1 & 0 & \cdots & 0 & \alpha^2 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 1 & \alpha^2 & \cdots & \alpha^2 & 0 & 0 & \cdots & \alpha^2 \\ \hline 0 & & & & & & & & & & & & & & & & & & \\ \vdots & & & & I_{r-1} & & & & & I_{r-1} & & & & & D_{r-2} & & & I_{r-2} \\ \hline 0 & & & & & & & & & & & & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right] \\
& \qquad \qquad \qquad 2.6.3: e \in E \\
& \left[\begin{array}{c|cccccccc} 1 & 0 & \cdots & 0 & \alpha^2 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 1 & \alpha^2 & \cdots & \alpha^2 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & & & & & & & \\ \vdots & & & & I_{r-1} & & & & & I_{r-1} & & & & & D_{r-2} & & & I_{r-2} \\ \hline 0 & & & & & & & & & & & & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right] \\
& \qquad \qquad \qquad 2.6.4: e \in F \\
& \left[\begin{array}{c|cccccccc} 1 & 0 & \cdots & 0 & \alpha^2 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & \alpha^2 & \cdots & \alpha^2 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & & & & & & & \\ \vdots & & & & I_{r-1} & & & & & I_{r-1} & & & & & D_{r-2} & & & I_{r-2} \\ \hline 0 & & & & & & & & & & & & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right]
\end{aligned}$$

Figure 2.6: Pivoted $GF(4)$ -matrices of T_r^2 for Sublemma 2.2.1.1

when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in \mathbf{F}$ Let e be the last element of F . We pivot by adding the last row to the second-last row, getting the matrix shown in Figure 2.6.4. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$. \square

Sublemma 2.2.1.2. *If M is isomorphic to G_r , then M' is isomorphic to G_{r-1} , T_{r-1}^2 , or T_{r-1}^1 .*

Subproof. Consider the $GF(4)$ matrix \mathfrak{M} for G_r in Figure 2.5.2. Let A be the set consisting of the first column of \mathfrak{M} . Let B be the set consisting of the second column of \mathfrak{M} . Let C , D , E , and F be the next four blocks of columns of \mathfrak{M} . For each of the next three columns, let the set consisting of this column of \mathfrak{M} be G , H , and I , respectively. Let J and K be the last two blocks of columns of \mathfrak{M} . Note that $|A| = |B| = |G| = |H| = |I| = 1$, $|C| = |D| = |E| = |J| = |K| = r - 2$, and $|F| = \binom{r-2}{2}$.

We now contract an element from each set. It is relatively easy to see that if X is a block, and $x, y \in X$ are elements of M , then $\text{si}(M/x) \cong \text{si}(M/y)$.

Let e be an element of $E(M)$.

$e \in \mathbf{A}$ In this case, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{B}$ In this case, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{C}$ Let e be the last element of C . We can see that $\text{si}(M/e) \cong G_{r-1}$.

$e \in \mathbf{D}$ Let e be the last element of D . We pivot by adding the last row to the second row, getting the matrix shown in Figure 2.7.1. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong G_{r-1}$.

2.7.1: $e \in D$

2.7.2: $e \in E$

2.7.3: $e \in F$

Figure 2.7: Pivoted $GF(4)$ -matrices of G_r for Sublemma 2.2.1.2 I

$e \in \mathbf{E}$ Let e be the last element of E . We pivot by adding the last row scaled by α to the second row, getting the matrix shown in Figure 2.7.2. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong G_{r-1}$.

$e \in \mathbf{F}$ Let e be the last element of F . We pivot by adding the last row to the second-last row, getting the matrix shown in Figure 2.7.3. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong G_{r-1}$.

$e \in \mathbf{G}$ We pivot by adding the first row to the second row, getting the matrix shown in Figure 2.8.1. Now, when we delete the first row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{H}$ We pivot by adding the first row scaled by α to the second row, getting the matrix shown in Figure 2.8.2. Now, when we delete the first row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in \mathbf{I}$ We pivot by adding the first row scaled by α^2 to the second row, getting the matrix shown in Figure 2.8.3. Now, when we delete the first row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in \mathbf{J}$ Let e be the last element of J . We pivot by adding the last row to the first row, getting the matrix shown in Figure 2.9.1. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong G_{r-1}$.

$e \in \mathbf{K}$ Let e be the last element of K . We pivot by adding the last row scaled by α to the first row, getting the matrix shown in Figure 2.9.2. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong G_{r-1}$. \square

Sublemma 2.2.1.3. *If M is isomorphic to HP_r , then M' is isomorphic to HP_{r-1} , T_{r-1}^2 , or T_{r-1}^1 .*

[illegible]

Figure 2.8: Pivoted $GF(4)$ -matrices of G_r for Sublemma 2.2.1.2 II

[illegible]

Figure 2.9: Pivoted $GF(4)$ -matrices of G_r for Sublemma 2.2.1.2 III

Subproof. Consider the $GF(4)$ matrix \mathfrak{M} for HP_r in Figure 2.5.3. Let A be the set consisting of the first column of \mathfrak{M} . Let B be the set consisting of the second column of \mathfrak{M} . Let C , D , E , and F be the next four blocks of columns of \mathfrak{M} . For each of the next three columns, let the set consisting of this column of \mathfrak{M} be G , H , and I , respectively. Let J and K be the last two blocks of columns of \mathfrak{M} . Note that $|A| = |B| = |G| = |H| = |I| = 1$, $|C| = |D| = |E| = |J| = |K| = r - 2$, and $|F| = \binom{r-2}{2}$.

We now contract an element from each set. It is relatively easy to see that if X is a block, and $x, y \in X$ are elements of M , then $\text{si}(M/x) \cong \text{si}(M/y)$.

Let e be an element of $E(M)$.

$e \in A$ In this case, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in B$ In this case, we can see that $\text{si}(M/e) \cong T_{r-1}^2$.

$e \in C$ Let e be the last element of C . We can see that $\text{si}(M/e) \cong HP_{r-1}$.

$e \in D$ Let e be the last element of D . We pivot by adding the last row to the second row, getting the matrix shown in Figure 2.10.1. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong HP_{r-1}$.

$e \in E$ Let e be the last element of E . We pivot by adding the last row scaled by α to the first and second rows, getting the matrix shown in Figure 2.10.2. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong HP_{r-1}$.

$e \in F$ Let e be the last element of F . We pivot by adding the last row to the second-last row, getting the matrix shown in Figure 2.10.3. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong HP_{r-1}$.

$e \in G$ We pivot by adding the first row to the second row, getting the matrix shown in Figure 2.11.1. Now, when we delete the first row and suppress

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha & 0 & \cdots & \alpha^2 & \cdots & \alpha^2 \\ 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & \alpha & \cdots & \alpha & \alpha^2 & 0 & \cdots & \alpha^2 \\ & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & & \end{bmatrix}$$

[illegible]

[illegible]

Figure 2.10: Pivoted $GF(4)$ -matrices of HP_r for Sublemma 2.2.1.3 I

[illegible]

parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{H}$ We pivot by adding the first row scaled by α to the second row, getting the matrix shown in Figure 2.11.2. Now, when we delete the first row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{I}$ We pivot by adding the first row scaled by α^2 to the second row, getting the matrix shown in Figure 2.11.3. Now, when we delete the first row and suppress parallel elements, we can see that $\text{si}(M/e) \cong T_{r-1}^1$.

$e \in \mathbf{J}$ Let e be the last element of J . We pivot by adding the last row scaled by α to the first row, getting the matrix shown in Figure 2.12.1. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong HP_{r-1}$.

$e \in \mathbf{K}$ Let e be the last element of K . We pivot by adding the last row scaled by α^2 to the first row, and then adding the last row to the second row, getting the matrix shown in Figure 2.12.2. Now, when we delete the last row and suppress parallel elements, we can see that $\text{si}(M/e) \cong HP_{r-1}$. \square

When we merge Sublemmas 2.2.1.1, 2.2.1.2, and 2.2.1.3, we get the desired result. \square

We are interested in the cases where $e \in E(M)$ is not in any five-point lines. The following corollary gives this characterisation.

Corollary 2.2.2. *Let M be a member of \mathcal{G}_r . Let M' be isomorphic to $\text{si}(M/e)$ for some $e \in E(M)$ such that e is not in any five-point lines.*

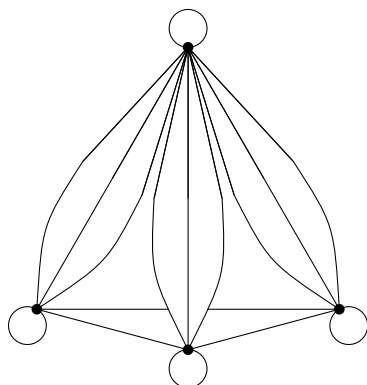
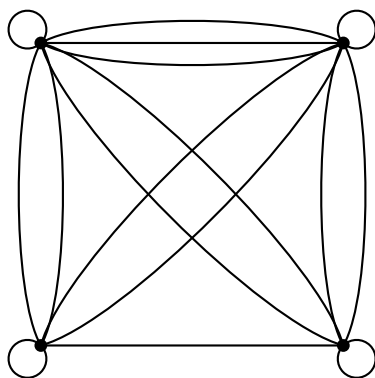
- (i) *If M is isomorphic to T_r^2 , then M' is isomorphic to T_{r-1}^2 .*
- (ii) *If M is isomorphic to G_r , then M' is isomorphic to G_{r-1} .*
- (iii) *If M is isomorphic to HP_r , then M' is isomorphic to HP_{r-1} .*

2.2.4 \mathbb{G} -graphic representations

Recall that a \mathbb{G} -matroid M is \mathbb{G} -graphic if there exists a \mathbb{G} -representation A of M such that A has at most two non-zero entries in each column.

[illegible]

Figure 2.12: Pivoted $GF(4)$ -matrices of HP_r for Sublemma 2.2.1.3 III

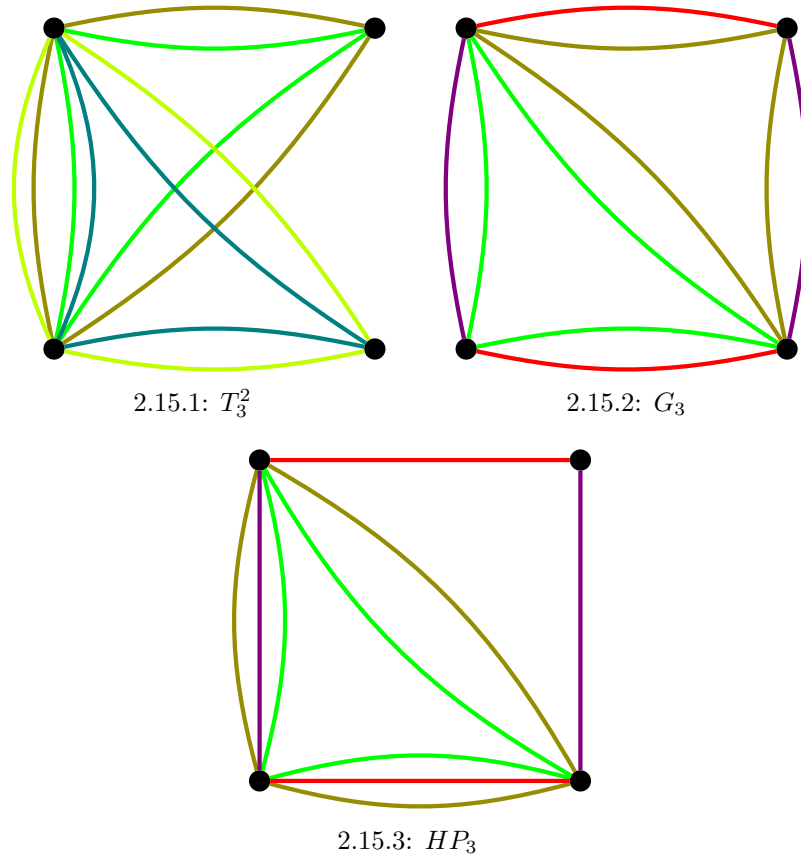
Figure 2.13: \mathbb{G} -graphic representation of T_4^2 Figure 2.14: \mathbb{G} -graphic representation of G_4

Two of the families, T_r^2 and G_r are \mathbb{G} -graphic matroids. The third family, HP_r , is not \mathbb{G} -graphic. We prove Conjecture 1.2.3 for the class of \mathbb{G} -graphic matroids in Chapter 5.

To construct a \mathbb{G} -graphic representation of T_r^2 , take a collection of r vertices, each with a loop. Select $r - 1$ of these vertices and construct K_{r-1} . Finally, construct a parallel class of size three from the unselected vertex to each vertex in the K_{r-1} . A \mathbb{G} -graphic representation of T_4^2 is given in Figure 2.13.

To construct a \mathbb{G} -graphic representation of G_r , take a collection of r vertices, each with a loop. Select $r - 2$ of these vertices and construct K_{r-2} . Between the two unselected vertices u and v , construct a parallel class of size three. Finally, construct a parallel class of size two from each of u and v to each vertex in the K_{r-2} . A \mathbb{G} -graphic representation of G_4 is given in Figure 2.14.

Note that \mathbb{G} -graphic representations are typically weighted and directed. However, we have left this information out.

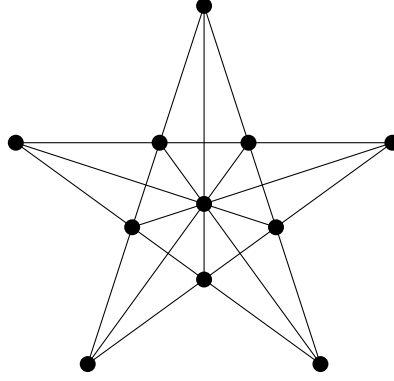
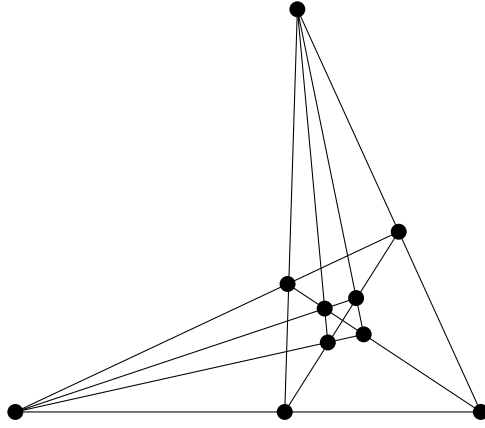
Figure 2.15: Augmented clique representations of all members of \mathcal{G}_3

2.2.5 Augmented Cliques

From looking at the matrices of the three families in Figure 2.1, it is easy to see that each of the families contains a spanning clique. In Chapter 6 we will show that there are two ways to extend from a spanning clique in a golden-mean way, called green triangles and red matchings. We collect this information in a graphical form, and construct augmented cliques. This construction is detailed in Section 6.2. Augmented clique representations of all members of \mathcal{G}_3 are given in Figure 2.15.

2.3 Rank Three

In various places, we will need to know all rank-three golden-mean matroids. To this end, we have the following lemma. Note that this lemma also confirms

Figure 2.16: The Betsy Ross (BR)Figure 2.17: Y_{10}

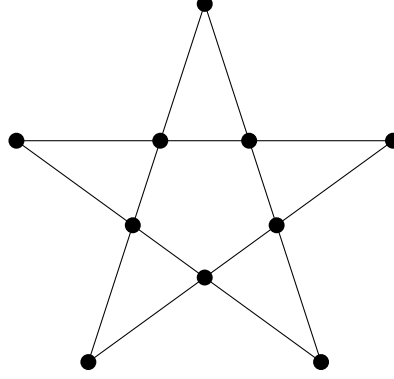
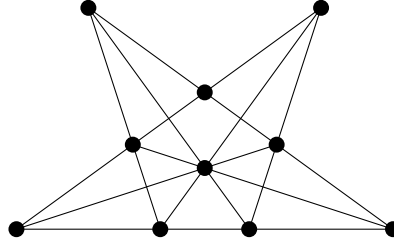
Conjecture 1.2.3 for rank three.

Lemma 2.3.1. *Let M be a simple golden-mean matroid of rank three. Then M is isomorphic to a restriction of one of the following matroids:*

- BR , The Betsy Ross (Figure 2.16),
- T_3^2 (Figure 2.2),
- G_3 (Figure 2.3),
- HP_3 (Figure 2.4), or
- Y_{10} (Figure 2.17).

Proof. We prove this using a computer search, which is detailed in Appendix 2.A.4. This also follows from independent work by Archer [1], Hliněný (personal correspondence), and Pendavingh (personal correspondence). \square

There are two ten-element restrictions of BR . These are S_{10} and $BR \setminus p$. A geometric representation of S_{10} is given in Figure 2.18 and a geometric

Figure 2.18: S_{10} Figure 2.19: $BR \setminus p$

representation of $BR \setminus p$ is given in Figure 2.19.

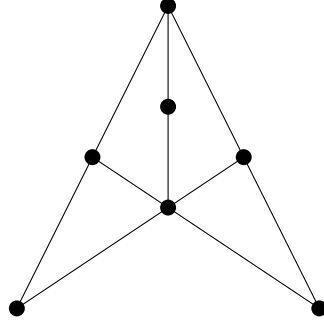
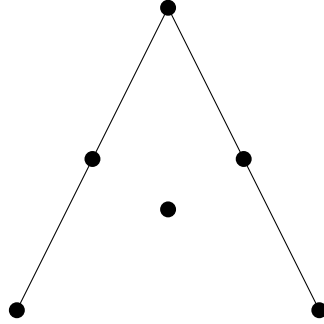
2.4 2-rounded Sets

We shall say that a matroid M **uses** an element e or a set Z if $e \in E(M)$ or $Z \subseteq E(M)$.

Definition 2.4.1 (Seymour, in Oxley [10, Page 481]). A set \mathcal{M} of matroids is **2-rounded** if every member of \mathcal{M} is 3-connected and the following condition holds: If M is a 3-connected matroid with a minor in \mathcal{M} and Z is a 2-element subset of $E(M)$, then M has a minor from \mathcal{M} using Z .

We will use the following results at various locations in this thesis.

Lemma 2.4.2 (Seymour, Theorem 3.1 [18]). *If x and y are elements of a non-binary 3-connected matroid M , then M has a $U_{2,4}$ minor using both x and y .*

Figure 2.20: $F_7^=$ Figure 2.21: Q_6

Lemma 2.4.3. *Within the class of golden-mean but non-ternary matroids, the set $\{U_{2,5}, U_{3,5}, F_7^=, (F_7^=)^*, Q_6\}$ is a 2-rounded set of matroids.*

Proof. We prove this by a computer check with Sage, using [10, Theorem 12.3.9]. The code is in Appendix 2.A.5 □

Geometric representation of the matroids $F_7^=$ and Q_6 are given in Figure 2.20 and Figure 2.21, respectively.

Appendix 2.A Code

All code is written in Python for Sage [19], in particular the matroids package [14]. In order to get it to run, one may need to import the advanced matroid functions: `from sage.matroids.advanced import *`.

2.A.1 Universal Variables

```
funds = [0, 1, 4, 5, 6, 14, 15, 16]
```

The allowed cross-ratios for golden-mean $GF(19)$ matrices are stored in `funds`.

2.A.2 Universal Functions

```
def is_gm(M):
    if M.cross_ratios().issubset(funds):
        return True
    else:
        return False
```

This function, `is_gm()`, returns `True` if the given $GF(19)$ -matrix represents a golden-mean matroid and `False` otherwise.

```
def matroid_set_intersection(one, two):
    OutSet = []
    for M in one:
        seen = False
        for N in two:
            if N.is_isomorphic(M):
                OutSet.append(M)
                break
    return OutSet
```

This function, `matroid_set_intersection()`, takes two lists, `one` and `two`, of matroids and returns the matroids from `one` that are isomorphic to a matroid in `two`.

2.A.3 Lemma 2.1.11

```
# set up tau
t = 1/2*sqrt(5) + 1/2

# set up the GM-set
gm = [0,1,-1]
for i in range(1,5): # should be high enough
    gm.append(t^i)
```

```

gm.append(t^(-i))
gm.append(-t^i)
gm.append(-t^(-i))

# and the fundamentals
test = [[t, 5], [-t, 14], [1/t, 4], [-1/t, 15], [t^2, 6], [1/(t
    ^2), 16]]

runs = []
for xy in Tuples(test, 2):
    M = Matroid(matrix(GF(19), [[1,0,1,1,1],[0,1,1,xy[0][1], xy
        [1][1]]]))
    if is_gm(M):
        runs.append(xy[0][0] - xy[1][0])

# make sure they're all in gm
all([d in gm for d in runs])

```

This code starts by constructing $\{\pm\tau^i \mid i \in \mathbb{Z}\}$ for $i \in \{-5, -4, \dots, 4, 5\}$, and storing these values in `gm`. The list `test` contains tuples of the form $(\psi(x), x)$, for relevant x . These values come from Table 2.1. We then form \mathfrak{N} over $GF(19)$, and find out which values of p and q give us a matroid M with cross-ratios in S . We then compute $\psi(p) - \psi(q)$ and check that this is a member of `gm`, proving the claim in the lemma.

2.A.4 Lemma 2.3.1

```

start_matrix = identity_matrix(GF(19), 3)
M = Matroid(start_matrix)
MS = [M]
count = 0
big = []
while len(MS) > 0:
    MS_hold = []
    for N in MS:
        exts = N.linear_extensions(simple=True, fundamentals=funds)

```

```

    if exts == []:
        big.append(N)
    for 0 in exts:
        MS_hold.append(0)
MS = MS_hold
count = count + 1
print [count, len(MS), len(big)]

print "now I need to isomorphism"
big = get_nonisomorphic_matroids(big)
print "finished"

```

This code starts from an identity matrix and builds all simple golden-mean matroids (over $GF(19)$) of rank three. If a matroid appears that has no extensions, then this matroid is extremal, and we store it in the **big** list. Once there are no more possible extensions, we have stored all extremal rank-three golden-mean matroids in **big**, so we get one copy of each isomorphism class.

Upon running this code (which takes a long time), we discover the matroids listed in Lemma 2.3.1.

2.A.5 Lemma 2.4.3

```

# set up the matroids
F7ECC = {2: ['abc', 'aef', 'cde', 'beg', 'cfg'], 3: ['abcdefg'
    ]}
U25 = matroids.Uniform(2,5)
Q6 = matroids.named_matroids.Q6()
F7E = Matroid(groundset='abcdefg', circuit_closures = F7ECC)

# and the duals (except Q6)
U25d = U25.dual()
F7Ed = F7E.dual()
ThreeRounded=[U25,U25d,Q6,F7E,F7Ed]

```

```

# since we don't have unique representation, get all the
  representations.

RoundList = []
for vect in Tuples(funds, 2):
    reduced_matrix = matrix(GF(19), [[1, 1, 1], [1, vect[0], vect
        [1]]])
    M = Matroid(reduced_matrix=reduced_matrix)
    if M.is_isomorphic(U25):
        if is_gm(M):
            if vect[0] <= vect[1]:
                RoundList.append(M)
                RoundList.append(M.dual())
for vect in Tuples(funds, 2):
    reduced_matrix = matrix(GF(19), [[1, 0, 1], [1, 1, vect[0]],
        [0, 1, vect[1]]])
    M = Matroid(reduced_matrix=reduced_matrix)
    if M.is_isomorphic(Q6):
        if is_gm(M):
            RoundList.append(M) # no dual as Q6 is self-dual
for vect in Tuples(funds, 1):
    reduced_matrix = matrix(GF(19), [[1, 1, 1, 0], [1, 0, 1, 1],
        [0, 1, 1, vect[0]]])
    M = Matroid(reduced_matrix=reduced_matrix)
    if M.is_isomorphic(F7E):
        if is_gm(M):
            RoundList.append(M)
            RoundList.append(M.dual())

for L in RoundList:
    for M in L.linear_extensions(simple=True, fundamentals=funds)
        :
    E = M.groundset()
    for x in E:
        for y in E.difference([x]):

```

```

    BadPair = True
    for N in RoundList:
        SizeDiff = M.size() - N.size()
        RankDiff = M.rank() - N.rank()
        for F in M.flats(RankDiff):
            if len(F) <= SizeDiff and F.isdisjoint([x,y]):
                for X in Subsets(E.difference(F).difference([x,y]), SizeDiff - len(F)):
                    if M.contract(F).delete(X).is_isomorphic(N):
                        BadPair=False
        if BadPair:
            print(M.circuit_closures(), x, y)
    for M in L.linear_coextensions(cosimple=True, fundamentals=funds):
        E = M.groundset()
        for x in E:
            for y in E.difference([x]):
                BadPair = True
                for N in RoundList:
                    SizeDiff = M.size() - N.size()
                    RankDiff = M.rank() - N.rank()
                    for F in M.flats(RankDiff):
                        if len(F) <= SizeDiff and F.isdisjoint([x,y]):
                            for X in Subsets(E.difference(F).difference([x,y]), SizeDiff - len(F)):
                                if M.contract(F).delete(X).is_isomorphic(N):
                                    BadPair=False
                if BadPair:
                    print(M.circuit_closures(), x, y)

```

This code proves Lemma 2.4.3. It starts by constructing all possible $GF(19)$ -representations of $U_{2,5}$, Q_6 , and F_7^\perp , and, where appropriate, their duals. We then construct all single-element extensions of these matroids, and try to place all subsets of size two into one of these matroids.

If you plug in a number and the math starts getting creepy (anything involving fractions or negative numbers is creepy)...

Doug Pierce

3

Connectivity

In this chapter we prove that any minimal counterexample to Conjecture 1.2.3 without a BR minor is vertically 4-connected. We will use this result later in the thesis.

3.1 3-connectivity

Definition 3.1.1. Let $M = (E, r)$ be a matroid and let $k > 1$ be an integer. A **k -separation** of M is a partition (X, Y) of E with the property that $|X|, |Y| \geq k$, and $r(X) + r(Y) - r(M) < k$. The separation is an **exact k -separation** if $r(X) + r(Y) - r(M) = k - 1$. If M has no n -separations for all $n \leq k$, then M is **$(k+1)$ -connected**.

Recall the function $h(r)$ from Section 1.2:

$$h(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Lemma 3.1.2. *Let M be a rank- r simple golden-mean matroid such that $\epsilon(M) \geq h(r)$. If all proper minors N of M have the property that $\epsilon(N) \leq h(r)$, then M is 2-connected.*

Proof. If M has rank three, then, by Lemma 2.3.1, this is obviously true. Hence M must have rank at least four.

Assume that M is not 2-connected. Then there exists an exact 1-separation (X_1, X_2) of M . Let $r(X_i) = r_i$ for $i \in \{1, 2\}$. As $M|X_i$ is a proper minor of M , $\epsilon(M|X_i)$ can be no larger than $\binom{r_i+3}{2} - 4$, for $i \in \{1, 2\}$. Note that this is not the definition of $h(r)$, as we need to consider the case where $r_i = 3$. So

$$\begin{aligned} \epsilon(M) &= \epsilon(M|X_1) + \epsilon(M|X_2) \\ &\leq \binom{r_1+3}{2} + \binom{r_2+3}{2} - 8 \\ &= \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 - 4). \end{aligned} \quad (3.1.1)$$

Also,

$$\begin{aligned} \epsilon(M) &\geq \binom{r+3}{2} - 5 \\ &= \binom{r_1+r_2+3}{2} - 5 \\ &= \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 + 2r_1r_2 - 4). \end{aligned} \quad (3.1.2)$$

Combining (3.1.1) and (3.1.2), we get

$$\begin{aligned} \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 - 4) &\geq \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 + 2r_1r_2 - 4) \\ 0 &\geq 2r_1r_2. \end{aligned}$$

As both r_1 and r_2 are positive, this is a contradiction. Therefore M is 2-connected. \square

Lemma 3.1.3. *Let M be a rank- r simple golden-mean matroid such that $\epsilon(M) \geq h(r)$. If all proper minors N of M have the property that $\epsilon(N) \leq h(r)$, then M is 3-connected.*

Proof. If M has rank three, then, by Lemma 2.3.1, this is obviously true. Hence M must have rank at least four.

Assume that M is not 3-connected. Then, by Lemma 3.1.2, M is 2-connected, so there are no exact 1-separations. Hence there exists an exact 2-separation (X_1, X_2) of M . Let $r(X_i) = r_i$. As $M_i = M|X_i$ is a proper minor of M , it follows that $\epsilon(M_i)$ can be no larger than $\binom{r_i+3}{2} - 4$, for $i \in \{1, 2\}$. Note that this is not the definition of $h(r)$, as we need to consider the case where $r_i = 3$. So

$$\begin{aligned} \epsilon(M) &= \epsilon(M_1) + \epsilon(M_2) \\ &\leq \binom{r_1+3}{2} + \binom{r_2+3}{2} - 8 \\ &= \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 - 4). \end{aligned} \quad (3.1.3)$$

Also,

$$\begin{aligned} \epsilon(M) &\geq \binom{r+3}{2} - 5 \\ &= \binom{(r_1 + r_2 - 1) + 3}{2} - 5 \\ &= \frac{1}{2} (r_1^2 + r_2^2 + 3r_1 + 3r_2 + 2r_1r_2 - 8). \end{aligned} \quad (3.1.4)$$

Combining (3.1.3) and (3.1.4), we get

$$\begin{aligned} \frac{1}{2} (r_1^2 + r_2^2 + 5r_1 + 5r_2 - 4) &\geq \frac{1}{2} (r_1^2 + r_2^2 + 3r_1 + 3r_2 + 2r_1r_2 - 8) \\ 2r_1 + 2r_2 &\geq 2r_1r_2 - 4 \\ r_1 + r_2 &\geq r_1r_2 - 2 \end{aligned}$$

If $r_i = 1$, then M contains a parallel class, and is therefore not simple, contradicting the definition of M .

If $r_1 = r_2 = 2$, then $r(M) = 3$, and this case has already been covered.

Hence, without loss of generality, $r_1 = 3$ and $r_2 = 2$, and $r(M) = 4$. Then M must have at least sixteen elements, and so $M_1 \cong BR$ and $M_2 \cong U_{2,5}$. If $E(M_1) \cap E(M_2)$ is empty, we would be able to find a $U_{2,6}$ -minor of M , and so would have at most $11 + 5 - 1 = 15$ elements in M , meaning $\epsilon(M) < h(r)$, contradicting the definition of M .

Therefore M is 3-connected. \square

3.2 Vertical Connectivity

We first define what it means for a matroid to be vertically 4-connected.

Definition 3.2.1. Let $M = (E, r)$ be a 3-connected matroid. The exact 3-separation (X, Y) of E is an **exact vertical 3-separation** if $r(X)$ and $r(Y)$ are both at least three.

Definition 3.2.2. Let $M = (E, r)$ be a 3-connected matroid. If every 3-separation (X, Y) of M has the property that $\min\{r(X), r(Y)\} \leq 2$, then M is **vertically 4-connected**.

The following result is well known. A proof is given here for completeness.

Lemma 3.2.3. *Let M be a vertically 4-connected matroid, and let $e \in E(M)$ be an element of M . Then $\text{si}(M/e)$ is 3-connected.*

Proof. First, we will show that $\text{si}(M/e)$ is 2-connected.

Sublemma 3.2.3.1. *The matroid $\text{si}(M/e)$ is 2-connected.*

Subproof. Assume that $\text{si}(M/e)$ is not 2-connected. Then there exists a 1-separation, (X_1'', X_2'') of $\text{si}(M/e)$. This induces a 1-separation, (X_1', X_2') of M/e . We now consider what happens to this partition in M . Let (X_1, X_2) be this partition in M . Without loss of generality, we can assume that

$e \in X_1$. Then $r_M(X_1) - r_{M/e}(X_1) = 1$, and $r_M(X_2) - r_{M/e}(X_2)$ is at most one. If $r_M(X_1) - r_{M/e}(X_1) = r_M(X_2) - r_{M/e}(X_2) = 1$, then

$$r_M(X_1) + r_M(X_2) - r_M(M) = 1.$$

Therefore (X_1, X_2) is a 2-separation of M , contradicting the fact that M is 3-connected. Hence $\text{si}(M/e)$ must be 2-connected. \square

Now assume that $\text{si}(M/e)$ is not 3-connected. Then there exists a 2-separation, (X_1'', X_2'') of $\text{si}(M/e)$. By putting back parallel elements and coloops, there is a 2-separation, (X_1', X_2') of M/e . We now consider what happens to this 2-separation in M . Let (X_1, X_2) be this 2-separation in M . If $r_M(X_1) = r_{M/e}(X_1') + 1$ and $r_M(X_2) = r_{M/e}(X_2')$, then (X_1, X_2) is a 2-separation of M , contradicting the fact that M is 3-connected. Hence $r_M(X_1) = r_{M/e}(X_1') + 1$ and $r_M(X_2) = r_{M/e}(X_2') + 1$. Then

$$\begin{aligned} r_M(X_1) + r_M(X_2) - r_M(M) &= r_{M/e}(X_1') + r_{M/e}(X_2') - (r_{M/e}(M/e) + 1) + 2 \\ &= 1 - 1 + 2 \\ &= 2. \end{aligned}$$

So (X_1, X_2) is a 3-separation of M . However, in $\text{si}(M/e)$, the rank of X_1'' and the rank of X_2'' are both at least two, as it is simple. So in M the rank of X_1 and the rank of X_2 are both at least three. This is a contradiction to M being vertically 4-connected, so $\text{si}(M/e)$ must be 3-connected. \square

For our applications we only need to consider golden-mean matroids with no BR minor.

Theorem 3.2.4. *Let M be a rank- r simple golden-mean matroid with no BR -minor such that $\epsilon(M) \geq \binom{r+3}{2} - 5$. If all proper minors N of M have the property that $\epsilon(N) \leq \binom{r(N)+3}{2} - 5$, then M is vertically 4-connected.*

Proof. Let $P = PG(r-1, 4)$. By Lemma 3.1.3 we know that M is 3-connected and ergo, if the lemma fails, M must have an exact vertical 3-separation (X_1, X_2) . View M as a restriction of P . Now,

$$r(\text{cl}_P(X_1) \cap \text{cl}_P(X_2)) \leq r(\text{cl}_P(X_1)) + r(\text{cl}_P(X_2)) - r(\text{cl}_P(X_1) \cup \text{cl}_P(X_2))$$

$$\begin{aligned}
&\leq r(X_1) + r(X_2) - r(X_1 \cup X_2) \\
&= r(X_1) + r(X_2) - r(M) \\
&= 2.
\end{aligned}$$

So the closures of X_1 and X_2 in P meet in a line L of P . Let $r_i = r(X_i)$ for each $i \in \{1, 2\}$. As (X_1, X_2) is a vertical 3-separation of M , both r_1 and r_2 must be at least three.

We consider $|L \cap E(M)|$, noting that it is at most five, as this is the maximum line length in a $GF(4)$ -representable matroid. The strategy of the proof is to consider, for each $i \in \{1, 2\}$, a simple rank- r_i minor M_i of M , obtained by deleting and contracting elements from the complement of X_i , that is spanned by X_i , contains $(X_1 \cup X_2) \cap L$, and has the maximum number of points among such minors. Thus, for $\{i, j\} = \{1, 2\}$, M_i is obtained from M by contracting elements in X_j so that as many points in X_j as possible are projected into the span of X_i . Clearly we may view M_i as a restriction of $P|(L \cup X_i)$.

Now

$$\begin{aligned}
\epsilon(M) &= |X_1| + |X_2| \\
&= (\epsilon(M_1) - |(E(M_1) \cap L) - X_1|) \\
&\quad + (\epsilon(M_2) - |(E(M_2) \cap L) - X_2|).
\end{aligned}$$

As M_i is a proper minor of M , $\epsilon(M_i)$ can be no larger than $\binom{r_i+3}{2} - 5$, for $i \in \{1, 2\}$. Thus

$$\begin{aligned}
\epsilon(M) &\leq \binom{r_1+3}{2} + \binom{r_2+3}{2} - 10 \\
&\quad - (|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|).
\end{aligned}$$

Also,

$$\epsilon(M) \geq \binom{(r_1 + r_2 - 2) + 3}{2} - 5.$$

So

$$\begin{aligned} \frac{1}{2}(r_1 + r_2)(r_1 + r_2 + 1) &\leq \frac{1}{2}((r_1 + 2)(r_1 + 3) + (r_2 + 2)(r_2 + 3)) - 5 \\ &\quad - (|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|). \end{aligned}$$

Expanding out gives

$$\begin{aligned} \frac{1}{2}(r_1^2 + r_2^2 + 2r_1r_2 + r_1 + r_2) &\leq \frac{1}{2}(r_1^2 + r_2^2 + 5r_1 + 5r_2 + 2) \\ &\quad - |(E(M_1) \cap L) - X_1| - |(E(M_2) \cap L) - X_2|. \end{aligned}$$

Hence

$$r_1r_2 - 2r_1 - 2r_2 - 1 \leq -(|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|).$$

And so

$$(r_1 - 2)(r_2 - 2) \leq 5 - (|E(M_1) \cap L| - |X_1 \cap L| + |E(M_2) \cap L| - |X_2 \cap L|). \quad (3.2.1)$$

But

$$\begin{aligned} |E(M_i) \cap L| &\geq |(X_1 \cup X_2) \cap L| \\ &= |X_1 \cap L| + |X_2 \cap L|, \end{aligned} \quad (3.2.2)$$

so, for each $i \in \{1, 2\}$,

$$(r_1 - 2)(r_2 - 2) \leq 5 - |E(M_i) \cap L|. \quad (3.2.3)$$

Next we take a basis B_1 for X_1 and extend it to a basis B for M . Then $|B - B_1| = r(M) - r(X_1) = r(X_2) - 2$. It follows that $r_{M/(B-B_1)}(X_2 - B) = 2$. This means that we can always project at least two points from X_2 into the span of X_1 by contracting only points in $X_2 - X_1$. Hence it can be assumed that M_1 satisfies

$$|E(M_1) \cap L| \geq 2. \quad (3.2.4)$$

Similarly,

$$|E(M_2) \cap L| \geq 2. \quad (3.2.5)$$

Combining (3.2.4) and (3.2.5) with (3.2.3), we get

$$(r_1 - 2)(r_2 - 2) \leq 3. \quad (3.2.6)$$

If r_1 and r_2 are both at least four, then (3.2.6) is a contradiction. Therefore, we can assume that $r_1 = 3$. So (3.2.6) becomes $r_2 \leq 5$.

Now suppose $|(X_1 \cup X_2) \cap L| \geq 3$.

Sublemma 3.2.4.1. *If $|(X_1 \cup X_2) \cap L| \geq 3$, then both M_1 and M_2 are 3-connected.*

Subproof. Let $M'_1 = M|(X_1 \cup (X_2 \cap L))$. Note that, by definition, X_1 spans L . Now

$$\begin{aligned} r(M'_1) &= r(M|(X_1 \cup (X_2 \cap L))) \\ &= r(M|X_1) \\ &= r(X_1). \end{aligned}$$

If (Y_1, Y_2) is a k -separation of M'_1 for some $k \leq 2$, then $r(Y_1) + r(Y_2) - r(X_1) \leq k - 1$, and, as $r(X_1) = r(M) - r(X_2) + 2$,

$$r(Y_1) + r(Y_2) - r(M) + r(X_2) - 2 \leq k - 1. \quad (3.2.7)$$

Without loss of generality, we may assume that $|Y_1 \cap L| \geq 2$. Then

$$\begin{aligned} r(Y_1 \cup X_2) &\leq r(\text{cl}(Y_1) \cup \text{cl}(X_2)) \\ &\leq r(\text{cl}(Y_1)) + r(\text{cl}(X_2)) - r(\text{cl}(Y_1) \cap \text{cl}(X_2)) \\ &\leq r(Y_1) + r(X_2) - r(\text{cl}(Y_1) \cap \text{cl}(X_2)). \end{aligned}$$

Observe that $\text{cl}(Y_1) \cap \text{cl}(X_2)$ contains L , so $r(\text{cl}(Y_1) \cap \text{cl}(X_2)) \geq 2$. Hence

$$r(Y_1 \cup X_2) \leq r(Y_1) + r(X_2) - 2. \quad (3.2.8)$$

Combining (3.2.7) with (3.2.8) gives $r(Y_2) + r(Y_1 \cup X_2) - r(M) \leq k - 1$, so $(Y_2, (Y_1 \cup X_2) - Y_2)$ is a k -separation of M , a contradiction. Thus M'_1 is 3-connected and, as M_1 is obtained from M'_1 by adding elements that are not loops, coloops or in parallel classes, M_1 is also 3-connected. Similarly, M_2 is 3-connected. \square

Sublemma 3.2.4.2. *If $|(X_1 \cup X_2) \cap L| \geq 3$, then M_1 and M_2 are not binary.*

Proof. If $|(X_1 \cup X_2) \cap L| > 3$, then it is easy to find a $U_{2,4}$ -minor of both M_1 and M_2 . Hence we can assume that $|(X_1 \cup X_2) \cap L| = 3$. From (3.2.6) we know that $r_1 = 3$ and $r_2 \leq 5$.

Firstly, assume that M_1 is binary. Then, as M_1 is golden-mean, it is representable over $GF(5)$, and hence, by Theorem 1.1.8, it is regular. Therefore, by Theorem 1.1.9, $\epsilon(M_1) \leq 6$. If $r_2 = 3$, then $r = 4$, and so $\epsilon(M) \geq 16$. Hence $\epsilon(M_2) \geq 13$, contradicting Lemma 2.3.1. Likewise, if $r_2 = 4$, then $\epsilon(M_2) \geq 20$, and if $r_2 = 5$, then $\epsilon(M) \geq 28$, both contradicting the definition of M . Hence M_1 cannot be binary.

Now assume that M_2 is binary. As M_1 is a rank-three proper minor of M , by the definition of M , $\epsilon(M_1) \leq 10$. Now consider M_2 . As it is binary and representable over $GF(5)$, it is therefore regular by Theorem 1.1.8. If $r_2 = 3$, then $r = 4$, and so $\epsilon(M) \geq 16$. Hence $\epsilon(M_2) \geq 9$. Likewise, if $r_2 = 4$, then $\epsilon(M_2) \geq 16$, and if $r_2 = 5$, then $\epsilon(M_2) \geq 24$. In all three cases, Theorem 1.1.9 is contradicted, and hence M_2 cannot be binary. \square

Sublemma 3.2.4.3. *If $|(X_1 \cup X_2) \cap L| \geq 3$, then $|(X_1 \cup X_2) \cap L| \geq 4$, and $r_1 = r_2 = 3$.*

Subproof. By Sublemma 3.2.4.2, M_i is not binary. Hence by Lemma 2.4.2, for $\{i, j\} = \{1, 2\}$, the matroid M_i has a $U_{2,4}$ minor using $(X_1 \cup X_2) \cap L$, and so $|E(M_j) \cap L| \geq 4$.

Firstly, assume that $|E(M_1) \cap L| = |E(M_2) \cap L| = 4$. Then (3.2.1) becomes

$$r_2 \leq 7 + |X_1 \cap L| + |X_2 \cap L| - 4 - 4$$

$$= |X_1 \cap L| + |X_2 \cap L| - 1. \quad (3.2.9)$$

If $|X_1 \cap L| + |X_2 \cap L| < 4$, then (3.2.9) becomes $r_2 \leq 2$, contradicting the fact that (X_1, X_2) is a vertical 3-separation of M .

If $|X_1 \cap L| + |X_2 \cap L| = 5$, then $|E(M_1) \cap L| = |E(M_2) \cap L| = 5$, and (3.2.3) implies that $r_1, r_2 \leq 2$, contradicting the fact that (X_1, X_2) is a vertical 3-separation of M .

Hence $|X_1 \cap L| + |X_2 \cap L| = 4$, and (3.2.9) becomes $r_2 \leq 3$, and the fact that (X_1, X_2) is a vertical 3-separation of M implies that $r_2 = 3$.

Next, assume that $|E(M_1) \cap L| = 4$ and $|E(M_2) \cap L| = 5$. Then (3.2.1) becomes

$$\begin{aligned} r_2 &\leq 7 + |X_1 \cap L| + |X_2 \cap L| - 4 - 5 \\ &= |X_1 \cap L| + |X_2 \cap L| - 2. \end{aligned} \quad (3.2.10)$$

Because L is a line of the projective geometry $PG(r-1, 4)$, it contains at most five elements. Hence $|X_1 \cap L| + |X_2 \cap L| \leq 5$, and so (3.2.10) becomes $r_2 \leq 3$, and the fact that (X_1, X_2) is a vertical 3-separation of M implies that $r_2 = 3$.

Lastly, assume that $|E(M_1) \cap L| = |E(M_2) \cap L| = 5$. Then (3.2.1) becomes

$$\begin{aligned} r_2 &\leq 7 + |X_1 \cap L| + |X_2 \cap L| - 5 - 5 \\ &= |X_1 \cap L| + |X_2 \cap L| - 3. \end{aligned} \quad (3.2.11)$$

As L is a line of the projective geometry $PG(r-1, 4)$, it contains exactly five elements. Hence $|X_1 \cap L| + |X_2 \cap L| \leq 5$, and so (3.2.11) becomes $r_2 \leq 2$, contradicting the fact that (X_1, X_2) is a vertical 3-separation of M .

So $3 \leq |X_1 \cap L| + |X_2 \cap L| = |(X_1 \cup X_2) \cap L| \leq 4$, and $r_1 = r_2 = 3$. \square

We will now show that $|(X_1 \cup X_2) \cap L|$ can be neither three nor four. Note that (3.2.3) implies that $|(X_1 \cup X_2) \cap L| \neq 5$.

Sublemma 3.2.4.4. $|(X_1 \cup X_2) \cap L| \neq 4$.

Subproof. If $|(X_1 \cup X_2) \cap L| = 4$, then, as $M|X_i$ is a proper rank-three minor of M , it has at most ten elements. Furthermore, as M has rank four, $\epsilon(M) \geq 16$. Hence $|X_1 - L| = |X_2 - L| = 6$.

Therefore, for $i \in 1, 2$, we can see that $\epsilon(M|(X_i \cup L)) = 10$. So, by Lemma 2.3.1, $M|(X_i \cup L)$ is isomorphic to one of HP_3 , $BR \setminus p$, Y_{10} , S_{10} , T_3^2 , or G_3 .

However, the matroids HP_3 and T_3^2 have no line of exactly four points, so it is not possible for either of them to be one of the $M|(X_i \cup L)$ restrictions. Hence M is two (not necessarily distinct) matroids, N_1 and N_2 , from the set $\{S_{10}, BR \setminus p, Y_{10}, G_3\}$ identified along a four-point line. No matter what N_1 is, it is easy to see that an extra point can be projected onto L , meaning that $|E(M_2) \cap L| = 5$, contradicting (3.2.3).

Hence $|(X_1 \cup X_2) \cap L| \neq 4$. \square

Sublemma 3.2.4.5. $|(X_1 \cup X_2) \cap L| \neq 3$.

Subproof. If $|(X_1 \cup X_2) \cap L| = 3$, then, as $M|X_i$ is a proper rank-three minor of M , it has at most ten elements. Furthermore, as M has rank four it must have at least sixteen elements. Hence we can assume that $|X_1 \cup L| = 7$ and $|X_2 \cup L| \geq 6$.

Therefore M is isomorphic to a member of $\{HP_3, T_3^2, G_3, S_{10}, BR \setminus p, Y_{10}\}$ identified along a three-point line with a matroid of at least 9 elements. In all cases, it is easy to see that we can contract two points from $M|(X_1 \cup L)$ onto L , giving $|E(M_2) \cap L| = 5$, contradicting (3.2.3). \square

Hence $|(X_1 \cup X_2) \cap L| \leq 2$. Then (3.2.1) becomes

$$3 \leq r_2 \leq 7 - |E(M_1) \cap L| - |E(M_2) \cap L| + |(X_1 \cup X_2) \cap L|. \quad (3.2.12)$$

We will now show that there is no possible value for $|(X_1 \cup X_2) \cap L|$.

Sublemma 3.2.4.6. $|(X_1 \cup X_2) \cap L| \neq 0$.

Subproof. Assume that $|(X_1 \cup X_2) \cap L| = 0$. From (3.2.4) and (3.2.5) we know that both $|E(M_1) \cap L|$ and $|E(M_2) \cap L|$ are at least two. Combining this information with (3.2.12), we see that $r_2 = 3$ and both $|E(M_1) \cap L|$ and

$|E(M_2) \cap L|$ must be exactly two. By assumption, $\epsilon(M) \geq 16$, so, without loss of generality, $|X_2| \geq 8$. As $M|(X_2)$ is $GF(5)$ -representable, if $M|(X_2)$ has no $U_{2,4}$ -minor, then it is regular, and so it must be no larger than $M(K_4)$. If $M|(X_2)$ has a $U_{2,4}$ -minor, then it is possible to contract a point from $M|(X_2)$ and put four points on L , so $|E(M_1) \cap L| = 4$, which is a contradiction to $|E(M_1) \cap L|$ being equal to two. Hence $\epsilon(M|(X_2)) \leq 6$, a contradiction to $|X_2| \geq 8$. So $|(X_1 \cup X_2) \cap L|$ cannot equal zero. \square

Sublemma 3.2.4.7. $|(X_1 \cup X_2) \cap L| \neq 1$.

Subproof. Assume that $|(X_1 \cup X_2) \cap L| = 1$, and suppose that $r_2 = 3$. Then $r = 4$. Then, by (3.2.4), (3.2.5), and (3.2.12), we see that $2 \leq |E(M_i) \cap L| \leq 3$, for $i \in \{1, 2\}$. So, as M has rank four, it must have at least sixteen elements. So $|X_1 \cup X_2| \geq 16$. Hence, without loss of generality, $|X_1| \geq 8$. Now pick $x \in X_1 - L$. As $|E(M_2) \cap L| \leq 3$, there can be no more than three lines passing through x . Hence there must be at least one four-point line containing x . Now contract an element not on this line, giving $|E(M_2) \cap L| = 4$, a contradiction. So $r_2 \neq 3$. From (3.2.4) and (3.2.5), we know that both $|E(M_1) \cap L|$ and $|E(M_2) \cap L|$ are at least two. Combining this information with (3.2.12), we see that $r_2 = 4$, and $|E(M_i) \cap L| = 2$, for $i \in \{1, 2\}$. So $r = 5$, implying that M has at least 23 elements. So $|X_1 \cup X_2| \geq 23$. Also, $\epsilon(M|X_2)$ can be no larger than sixteen, so $|X_2| \leq 16$. Therefore $|X_1| \geq 7$. Now pick $x \in X_1 - L$. Because $|E(M_2) \cap L| = 2$, it follows that x is on at most two lines. Therefore, one of these lines has at least four points and we can contract an element on the other line to get $|E(M_2) \cap L| = 4$, a contradiction. So $|(X_1 \cup X_2) \cap L|$ cannot equal one. \square

Sublemma 3.2.4.8. $|(X_1 \cup X_2) \cap L| \neq 2$.

Subproof. Assume $|(X_1 \cup X_2) \cap L| = 2$. So (3.2.12) becomes

$$3 \leq r_2 \leq 9 - |E(M_1) \cap L| - |E(M_2) \cap L|. \quad (3.2.13)$$

Let $\{1, 2\} = \{i, j\}$. Assume that $|E(M_j) \cap L| = 2$, and let $E(M_j) \cap L = \{s, t\}$. Then pick $x \in X_i - L$. As $|E(M_j) \cap L| = 2$, everything in X_i must be on a line with x and either s or t . So $r_i = 3$. Hence r_j can be 3, 4, or 5. If $r_j = 3$, then, as M has rank four, $\epsilon(M) \geq 16$, so $|X_j \cup ((X_1 \cup X_2) \cap L)| \leq 10$,

and therefore $|X_i \cup ((X_1 \cup X_2) \cap L)| \geq 8$. Using the same reasoning, we see that if $r_j = 4$, then $|X_i \cup ((X_1 \cup X_2) \cap L)| \geq 9$, and if $r_j = 5$, then $|X_i \cup ((X_1 \cup X_2) \cap L)| \geq 10$. In all three cases, both lines through x must have at least three points on them, and we contract a point from $X_i - \{x, s, t\}$ to project at least three points onto L , implying that $|E(M_j) \cap L| \geq 3$, a contradiction.

So now it follows from (3.2.4), (3.2.5), and (3.2.13) that $|E(M_1) \cap L| = |E(M_2) \cap L| = 3$, and $r_2 = 3$. As M has rank four, $\epsilon(M) \geq 16$. Then, without loss of generality, $|X_1| \geq 8$. Pick $x \in X_1 - L$. Then, as $|E(M_2) \cap L| = 3$, the element x can be on at most three lines. However we place the remaining five points, we will always get one of these lines having at least four points, which can be projected onto L , implying that $|E(M_j) \cap L| \geq 4$, a contradiction.

Therefore $|(X_1 \cup X_2) \cap L| \neq 2$. \square

Hence, by Sublemmas 3.2.4.4 – 3.2.4.8, there are no possible values for $|(X_1 \cup X_2) \cap L|$, so our original assumption, that M has an exact vertical 3-separation, is incorrect. So M is vertically 4-connected. \square

*Be regular and orderly in your life, so that you may be violent
and original in your work.*

Gustave Flaubert

4

Lifts of Regular and Near-regular

4.1 Preliminaries

Let \mathcal{M}_1 be the set of golden-mean matroids M with the property that M/e is regular for some $e \in E(M)$. Let \mathcal{R} be the set of all minors of matroids in \mathcal{M}_1 . Note that M is in \mathcal{R} if and only if M is golden-mean, and there is some golden-mean single-element extension N of M , by the element e , such that N/e is regular. Equivalently, we might say that M is a golden-mean lift of a regular matroid.

Now let \mathcal{M}_2 be the set of golden-mean matroids M with the property that M/e is near-regular for some $e \in E(M)$. Let \mathcal{N} be the set of all minors of matroids in \mathcal{M}_2 . As above, note that M is in \mathcal{N} if and only if M is golden-mean, and there is some golden-mean single-element extension N of M , by the element e , such that N/e is near-regular. Equivalently, we might say that M is a golden-mean lift of a near-regular matroid. Also note that \mathcal{R} is a subclass of \mathcal{N} .

In this chapter, we prove that Conjecture 1.2.3 holds in the classes \mathcal{R} and \mathcal{N} .

Theorem 4.1.1. *Let \mathcal{R} be the class of matroids defined above. Then*

$$h_{\mathcal{R}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{R}$ is maximum-sized if and only if $M \cong T_{r(M)}^2$.

Theorem 4.1.2. *Let \mathcal{N} be the class of matroids defined above. Then*

$$h_{\mathcal{N}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{N}$ is maximum-sized if and only if M is isomorphic to a member of $\mathcal{G}_{r(M)}$ when $r(M) \neq 3$, or a member of $\mathcal{G}_3 \cup \{BR \setminus p\}$ when $r(M) = 3$.

4.1.1 Spikes

For various results, we need to understand the golden-mean spikes.

Definition 4.1.3 (Ding et al. [4]). For $n \geq 3$, a simple matroid M is a ***n-spike with tip t*** if it satisfies the following properties.

- (i) the ground set is the union of n lines, known as ***legs***, L_1, \dots, L_n , all having three points and passing through a common point t ;
- (ii) for all k in $\{1, 2, \dots, n-1\}$, the union of any k of L_1, \dots, L_n has rank $k+1$; and
- (iii) $r(L_1 \cup \dots \cup L_n) = n$.

We will refer to an n -spike with tip t as a ***n-spike***.

Let S be a n -spike with tip t representable over a field \mathbb{F} . If we choose a basis $\{1, \dots, n\}$ containing exactly one element from each of the lines L_i , then S can be represented in the form

$$\begin{array}{c}
\begin{array}{cccccc}
1 & 2 & 3 & \cdots & n & t
\end{array} \\
\left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & \cdots & 0 & 1 & x_1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 1 & x_2 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & x_3 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & x_n
\end{array} \right]
\end{array}$$

where x_1, \dots, x_n are elements of $\mathbb{F} \setminus \{1\}$ (Wu [25]).

We wish to characterise the golden-mean spikes. In particular, we are interested in the 4-spikes and 5-spikes.

Lemma 4.1.4. *There is only one golden-mean 4-spike.*

Proof. We prove this by an exhaustive computer search, the code for which is in Appendix 4.A.1. Upon running this code, we discover that up to isomorphism there is only one golden-mean 4-spike. This spike is represented over $GF(4)$ by the following matrix.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & \alpha & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \alpha
\end{bmatrix}$$

□

Lemma 4.1.5. *There are no golden-mean 5-spikes.*

Proof. We prove this by an exhaustive computer search, the code for which is in Appendix 4.A.2. Upon running this code, we discover that there are no golden-mean 5-spikes. This has also been independently verified by Van Zwam (personal correspondence). □

We want to know that it is not possible to extend a leg of the golden-mean 4-spike.

Lemma 4.1.6. *If M is a golden-mean 4-spike with tip t , $M' \setminus e = M$, where M' is golden-mean, and $\{e, a, b\}$ is a triangle, where $\{t, a, b\}$ is a leg of M , then $\{t, e\}$ is a circuit.*

Proof. Assume for contradiction that $\{e, t\}$ is not a circuit, so $\{e, t, a, b\}$ is a $U_{2,4}$ -restriction. Let M be the matroid represented over $GF(4)$ by the following matrix.

$$\begin{array}{c} t \\ \left[\begin{array}{ccccc} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & \alpha & 1 \\ b & 1 & 1 & 1 & \alpha \end{array} \right] \end{array}$$

As M is a spike, $\{t, a, b\}$ is a triangle. Hence, in M' , where $\{e, a, b\}$ is a triangle, $\{t, e, a, b\}$ must be dependent. Now consider $M' \setminus a = M''$. Note that $\{t, e, b\}$ is a triangle of M'' and that M'' is a 4-spike. By Lemma 4.1.4 there is only one golden-mean 4-spike, and so by Lemma 2.1.12 the column representing e is parallel to the column representing a . Hence $\{e, a\}$ is a circuit, a contradiction.

Therefore $\{t, e\}$ is a circuit. □

We rewrite Lemma 4.1.6 as the following Corollary.

Corollary 4.1.7. *If M is a simple golden-mean matroid, and M' is a 4-spike restriction of M with ℓ as a leg of M' , then ℓ is a flat in M .*

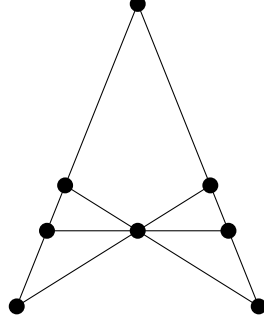
4.1.2 Small Circuits

The family of matroids T_r^1 are the near-regular matroids represented over \mathbb{U}_1 by the following families of matrices. A geometric representation of T_3^1 is given in Figure 4.1.

$$\left[\begin{array}{c|ccc|ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \beta & \cdots & \beta & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & \\ 0 & & & & & & & & & & & & \end{array} \right]$$

We also need to understand the simple regular and near-regular matroids with no circuits of size greater than four.

To that end, we need the following result.

Figure 4.1: T_3^1

Lemma 4.1.8 (Whittle [22, Section 5]). *Let M be a simple near-regular matroid of rank three. Then M is a restriction of T_3^1 (Figure 4.1).*

Let $K_{2,r}^\natural$ be the family of near-regular matroids represented by the following family of matrices.

$$\left[\begin{array}{c|ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \beta & \cdots & \beta \\ \hline 0 & & & & & & & & & \\ \vdots & & I_{r-1} & & & I_{r-1} & & & I_{r-1} & \\ \hline 0 & & & & & & & & & \end{array} \right]$$

Note that $K_{2,r}^\natural$ consists of $r - 1$ copies of $U_{2,4}$, known as legs, parallel-connected together at a point, known as the tip.

Lemma 4.1.9. *If M is a simple connected near-regular matroid with no circuit of size greater than four, then M is a restriction of either T_3^1 or $K_{2,r}^\natural$, for some r .*

Proof. We prove this by induction on $\epsilon(M)$.

If M has rank at most three then the result follows from Lemma 4.1.8. Hence, we can assume that the rank of M is at least four.

Firstly, assume that M is not 3-connected. Then, by Theorem 8.3.1 of Oxley [10], $M = M_X \oplus_2 M_Y$, for some matroids M_X and M_Y . Note that Theorem 8.3.1 of Oxley [10] also implies that $\epsilon(M_A) < \epsilon(M)$, for $A \in \{X, Y\}$.

Proposition 7.1.22 (ii) of Oxley [10] implies that both M_X and M_Y are connected. This means that we can apply the induction hypothesis to both $\text{si}(M_X)$ and $\text{si}(M_Y)$.

Now assume that b , the basepoint of the 2-sum $M_X \oplus_2 M_Y$, is in a circuit C_1 of size four in M_X . In M_Y , the element b is in a circuit C_2 with $|C_2| \geq 3$, since $r(M_Y) \geq 2$ (as otherwise $M = M_X \oplus_2 M_Y$ is just M_X with a parallel class). Now, by Proposition 7.1.20 of Oxley [10], $(C_1 - b) \cup (C_2 - b)$ is a circuit of M with at least five elements, a contradiction. Hence b is not in any circuits of size four, and so M_X is isomorphic to a restriction of $K_{2,r}^b$, where b is either the tip or parallel to the tip. A symmetric argument shows that M_Y is isomorphic to a restriction of $K_{2,r}^b$. If we 2-sum these matroids at the tip, we merely get a restriction of a larger $K_{2,r}^b$.

Now assume that M is 3-connected. Then $M \setminus e$ is connected for any element $e \in E(M)$ by Proposition 8.2.8 of Oxley [10], and hence, by induction, it must be isomorphic to a restriction of $K_{2,r}^b$ for some r .

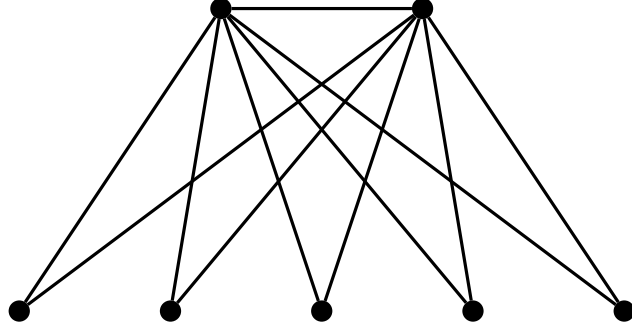
Consider $M \setminus e$. When we extend by e , M must be 3-connected, so we cannot add e parallel to the tip, as $K_{2,r}^b$ is not 3-connected. If e is in a triangle in M , then it is in the span of either one or two legs. As M has rank at least four, it has at least three legs, so there is a leg L of M such that L minus the tip is either a cocircuit of size at most two in M or a cocircuit-circuit of size three, both of which contradict 3-connectivity. Since we chose e arbitrarily, M has no triangles. Hence for all $e \in E(M)$, the matroid $M \setminus e \cong M(K_{2,r})$ for some r .

Let $\{x, y\}$ be a leg of $M \setminus e$. Then $\{e, x, y\}$ must be a cocircuit in M , as $\{x, y\}$ is a series pair. Hence $\{e, y\}$ must be a leg of $M \setminus x$. Let $\{u, v\}$ be some other leg of $M \setminus x$. Hence $\{e, u, v, y\}$ is a 4-circuit, so it meets every leg in M , as a leg with e added is a triad. This implies that there can be no more than three legs, so $r(M) \leq 4$, meaning that $r(M) = 4$ and hence $\epsilon(M) = 7$.

Now we see that $r(M^*) = 3$, and hence we can apply Lemma 4.1.8 to see that M is isomorphic to either P_7^* or O_7^* , as these are the only 3-connected seven element restrictions of $(T_3^1)^*$. Hence M has circuits of size greater than four, a contradiction. \square

When $b \geq 1$, the graph $K_{2,b}^+$ is the complete bipartite graph $K_{2,b}$ with the addition of an edge joining the partition of size two. A drawing of $K_{2,5}^+$ is given in Figure 4.2.

We now define some terminology related to $M(K_{2,b}^+)$. We will explore $M(K_{2,b}^+)$

Figure 4.2: $K_{2,5}^+$

more in Section 4.3.2.

Definition 4.1.10. Each three-point line in $M(K_{2,b}^+)$ is known as a **limb**. If $b = 1$, then the **head** of $M(K_{2,b}^+)$ is any element of $M(K_{2,b}^+)$. If $b > 1$, then the **head** of $M(K_{2,b}^+)$ is the element of $M(K_{2,b}^+)$ that is in b three-point lines.

By considering the regular restrictions of T_3^1 and $K_{2,r}^h$ we get the following corollary.

Corollary 4.1.11. *If M is a connected regular matroid and M has no circuit of size greater than four, then either $\text{si}(M) \cong M(K_4)$, or $\text{si}(M) \cong M(K_{2,b})$ for some b , or $\text{si}(M) \cong M(K_{2,b}^+)$ for some b .*

4.1.3 $\mathcal{L}(M, e)$

We often consider the matroid obtained by restricting to the long lines through e , contracting e and then simplifying. In this section, we develop this idea.

Definition 4.1.12. Let M be a matroid, and let e be an element of M , and let \mathfrak{L} be the set of long lines of M . Let $X = \{e\} \cup \{f \in E(M) \mid \exists L \in \mathfrak{L} \text{ with } e, f \in L\}$. Then $\mathcal{L}(M, e)$ is defined to be $\text{si}((M|X)/e)$.

Note that $\mathcal{L}(M, e)$ is isomorphic to a restriction of M/e . We wish to restrict the circuits contained in $\mathcal{L}(M, e)$.

Lemma 4.1.13. *Let $M = (E, r)$ be a maximum-sized golden-mean matroid, and let e be an element of M . If $\mathcal{L}(M, e)$ contains two circuits C_1 and C_2 , such that $|C_1| = |C_2| = 4$, then $|C_1 \cap C_2| \neq 3$.*

Proof. Assume that $|C_1 \cap C_2| = 3$. Then $C_1 = \{c_1, x, y, z\}$ and $C_2 = \{c_2, x, y, z\}$. As C_i for $i \in \{1, 2\}$ is from $\mathcal{L}(M, e)$, each element of C_i corresponds to a long line in M . Let these lines be l'_a for $a \in C_1 \cup C_2$. For each l'_a , let l_a be a three-point restriction of l'_a such that $e \in l_a$. Let

$$S_i = \bigcup_{a \in C_i} l_a$$

for $i \in \{1, 2\}$. We claim that S_1 and S_2 are 4-spikes.

To see this, note that $r(S_i) = 4$, and for all $k \in \{1, 2, 3\}$ the union of any k long lines from S_i has rank $k + 1$.

Let $N = M|(S_1 \cup S_2)$. So N consists of five long lines, in particular the long lines $\{e, c_1, c'_1\}$ and $\{e, c_2, c'_2\}$ are in N . Note that $N \setminus c_1 \setminus c'_1 = S_2$ and $N \setminus c_2 \setminus c'_2 = S_1$.

So $N \setminus c_2 \setminus c'_2$ has the $GF(4)$ representation

$$U = \begin{array}{c} \begin{array}{ccccccccc} & & c_1 & e & & & & c'_1 & \\ & & & & & & & & \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & a & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & b & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & c & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & d \end{array} \right] \end{array}$$

Furthermore, if we delete c_1 and c'_1 instead, we get

$$U' = \begin{array}{c} \begin{array}{ccccccccc} & & c_2 & e & & & & c'_2 & \\ & & & & & & & & \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & a & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & b & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & c & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & d' \end{array} \right] \end{array}$$

It follows from Lemma 4.1.4 that (a, b, c, d) and (a, b, c, d') are equal up to permutation. Hence $d = d'$.

As we have uniqueness of representations by Lemma 2.1.12, we can combine U and U' in the obvious way to see that c_1 and c_2 are parallel in N . This is a contradiction, so N cannot exist, and hence $|C_1 \cap C_2| \neq 3$. \square

Lemma 4.1.14. *Let M be a maximum-sized golden-mean matroid, and let*

e be a point of M . Then $\mathcal{L}(M, e)$ is near-regular.

Proof. If $\mathcal{L}(M, e)$ is not near-regular, then it must contain an excluded minor for the class of near-regular matroids, which are listed in [5, Theorem 1.2]. As $\mathcal{L}(M, e)$ is a minor of a golden-mean matroid, the excluded minor must be golden-mean. By considering the ten excluded minors for near-regular matroids, we see that only $U_{2,5}$ and $U_{3,5}$ are golden-mean.

If $\mathcal{L}(M, e)$ has a $U_{2,5}$ -minor, then we can find a rank-three restriction of M with eleven points and five copunctual lines, a contradiction to Lemma 2.3.1.

If $\mathcal{L}(M, e)$ has a $U_{3,5}$ -minor, then we can find two 4-circuits in $U_{3,5}$ that share three elements. This is a contradiction by Lemma 4.1.13, so $\mathcal{L}(M, e)$ can not have a $U_{3,5}$ -minor, and hence $\mathcal{L}(M, e)$ is near-regular. \square

We want to know exactly what $\mathcal{L}(M, e)$ consists of. The next two results provide this.

Lemma 4.1.15. $\mathcal{L}(M, e)$ has no circuits of size five or greater.

Proof. Assume that $\mathcal{L}(M, e)$ has a circuit $C = \{c_1, \dots, c_5\}$ of size five. By the definition of $\mathcal{L}(M, e)$, each c_i is in a parallel class of size at least two in M/e . Let $d_i \neq c_i$ be an element from the parallel class containing c_i , for $i \in 1, \dots, 5$. We claim that $S = M| \{e, c_1, \dots, c_5, d_1, \dots, d_5\}$ is a 5-spike.

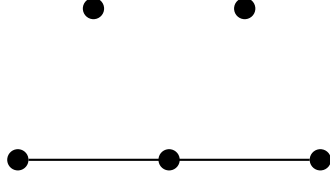
To see this, note that $r(S) = 5$, and that for all i , the three elements $\{e, c_i, d_i\}$ form a line L_i . Also, for all k in $\{1, 2, 3, 4\}$, the union of any k of L_1, \dots, L_5 has rank $k + 1$. Hence S is a 5-spike.

However, there are no golden-mean 5-spikes by Lemma 4.1.5, and so this is a contradiction. Hence $\mathcal{L}(M, e)$ can have no circuits of size five or greater. \square

Corollary 4.1.16. Let M be a maximum-sized golden mean matroid, and let e be a point of M . Then the connected components of $\mathcal{L}(M, e)$ are restrictions of either T_3^1 or $K_{2,x}^h$, for some x .

Proof. From Lemmas 4.1.14 and 4.1.15, we know that $\mathcal{L}(M, e)$ is a near-regular matroid with no circuits of size five or greater. The result now follows from Lemma 4.1.9. \square

Corollary 4.1.17. $\mathcal{L}(M, e)$ cannot have P_5 (Figure 4.3) as a minor.

Figure 4.3: P_5

Proof. This follows from Lemma 4.1.13. □

By considering the restrictions of T_3^1 and $K_{2,r}^1$ that do not have a P_5 -minor, we get the following result.

Corollary 4.1.18. *Let M be a maximum-sized golden mean matroid, and let e be a point of M . Then the connected components of $\mathcal{L}(M, e)$ are isomorphic to one of the following matroids:*

- (i) $U_{1,1}$,
- (ii) $U_{2,4}$,
- (iii) $M(K_4)$,
- (iv) $M(K_{2,b})$ for some $b \geq 2$, or
- (v) $M(K_{2,b}^+)$ for some $b \geq 1$.

4.1.4 Deficit Functions

As $\mathcal{L}(M, e)$ is the simplification of a contraction of a collection of long lines, we lose the length of each line upon this contraction. To regather this information into a useable format, we define the weight functions.

Henceforth, we are going to let M be a counterexample to either Theorem 4.1.1 or Theorem 4.1.2.

Function 4.1.19. The first function we are defining is the weight function w .

Let e be an element of the matroid M . If $i \in E(M) \setminus e$, then

$$w_i = |\text{cl}_M(\{e, i\})|.$$

If N is a component of $\mathcal{L}(M, e)$ (denoted by $N \triangleleft \mathcal{L}(M, e)$), then

$$w(N) = \sum_{i \in E(N)} (w_i - 2). \quad \diamond$$

By the definition of $\mathcal{L}(M, e)$, the minimum value of w_i is three. Also, as M is $GF(4)$ -representable, the maximum value is five. Hence for all $i \in E(\mathcal{L}(M, e))$,

$$3 \leq w_i \leq 5. \quad (4.1.1)$$

Let M have rank r . Then

$$\sum_{N \triangleleft \mathcal{L}(M, e)} r(N) = r(\mathcal{L}(M, e)) \leq r - 1. \quad (4.1.2)$$

We consider two possible options for M/e , and construct a function for each.

First, consider the case when M/e is regular. In this case, $\epsilon(M/e) \leq \binom{r}{2}$ by Theorem 1.1.9. Also, as M is a counterexample to Theorem 4.1.1, $\epsilon(M) \geq \binom{r+3}{2} - 5$.

Therefore

$$\epsilon(M) - \epsilon(M/e) \geq \binom{r+3}{2} - 5 - \binom{r}{2} = 3r - 2.$$

Hence

$$\sum_{N \triangleleft \mathcal{L}(M, e)} w(N) + 1 \geq 3r - 2. \quad (4.1.3)$$

Combining (4.1.2) with (4.1.3), we obtain

$$3 \sum_{N \triangleleft \mathcal{L}(M, e)} r(N) \leq \sum_{N \triangleleft \mathcal{L}(M, e)} w(N). \quad (4.1.4)$$

This motivates the second function.

Function 4.1.20. The next function we define is the regular deficit function d_r . If $N \triangleleft \mathcal{L}(M, e)$, then

$$d_r(N) = 3r(N) - w(N). \quad \diamond$$

Rewriting (4.1.4) using the regular deficit function, we get

$$\sum_{N \triangleleft \mathcal{L}(M, e)} d_r(N) \leq 0. \quad (4.1.5)$$

Now consider the case when M/e is near-regular. In this case, $\epsilon(M/e) \leq \binom{r+1}{2} - 2$ by Theorem 1.1.12. Also, as M is a counterexample to Theorem 4.1.2, $\epsilon(M) \geq \binom{r+3}{2} - 5$.

Therefore

$$\epsilon(M) - \epsilon(M/e) \geq \binom{r+3}{2} - 5 - \left(\binom{r+1}{2} + 2 \right) = 2r. \quad (4.1.6)$$

Hence

$$\sum_{N \triangleleft \mathcal{L}(M, e)} w(N) + 1 \geq 2r. \quad (4.1.7)$$

Combining (4.1.2) with (4.1.7), we obtain

$$2 \sum_{N \triangleleft \mathcal{L}(M, e)} r(N) + 1 \leq \sum_{N \triangleleft \mathcal{L}(M, e)} w(N). \quad (4.1.8)$$

This motivates the third function.

Function 4.1.21. The next function we define is the near-regular deficit function d_n . If $N \triangleleft \mathcal{L}(M, e)$, then

$$d_n(N) = 2r(N) - w(N). \quad \diamond$$

Rewriting (4.1.8) using the near-regular deficit function, we get

$$\sum_{N \triangleleft \mathcal{L}(M, e)} d_n(N) \leq -1. \quad (4.1.9)$$

We now go through each possible component of $\mathcal{L}(M, e)$ in turn and calculate the minimum value of both deficit functions. We do this by computing the maximum weight and then applying Functions 4.1.20 and 4.1.21.

Lemma 4.1.22. *Let N be a component of $\mathcal{L}(M, e)$ such that $N \cong M(K_{2,b})$ for some $b \geq 2$. Then $w_i = 3$ for all $i \in E(N)$, and hence the maximum value of $w(N)$ is $2b$.*

Proof. The matroid $M(K_{2,b})$ consists of b copies of $U_{2,3}$, known as legs, 2-summed at a point, and hence has rank $b + 1$. Each leg has two points on it, and any two legs, when considered independently, give a copy of $U_{3,4}$ in $\mathcal{L}(M, e)$, which leads to a 4-spike in M . It follows from Corollary 4.1.7 that the maximum size of the closure of the leg of a golden-mean 4-spike is three. Therefore from Function 4.1.19 we get that the maximum weight of $M(K_{2,b})$ is $2b$. \square

Corollary 4.1.23. *The minimum regular deficit of $M(K_{2,b})$ for some $b \geq 2$ is $b + 3$.*

Corollary 4.1.24. *The minimum near-regular deficit of $M(K_{2,b})$ for some $b \geq 2$ is two.*

Lemma 4.1.25. *Let N be a component of $\mathcal{L}(M, e)$ such that $N \cong M(K_4)$. Then $w_i = 3$ for all $i \in E(N)$, and hence the maximum value of $w(N)$ is six.*

Proof. The matroid $M(K_4)$ has six points and rank three. Each point, when considered in the correct set of four points, gives a copy of $U_{3,4}$ in $\mathcal{L}(M, e)$, which leads to a 4-spike in M . It follows from Corollary 4.1.7 that the maximum size of the closure of the leg of a golden-mean 4-spike is three. Therefore from Function 4.1.19 we get that the maximum weight of $M(K_4)$ is six. \square

Corollary 4.1.26. *The minimum regular deficit of $M(K_4)$ is three.*

Corollary 4.1.27. *The minimum near-regular deficit of $M(K_4)$ is zero.*

Lemma 4.1.28. *Let N be a component of $\mathcal{L}(M, e)$ such that $N \cong M(K_{2,b}^+)$ for some $b \geq 2$. Then $w_i = 3$ for all $i \in E(N) \setminus \{p\}$, where p is the head of N , and hence the maximum value of $w(N)$ is $2b + 3$.*

Proof. Recall that the matroid $M(K_{2,b}^+)$ is the matroid $M(K_{2,b})$ with the addition of the basepoint p , known as the head of $M(K_{2,b}^+)$, from the 2-sum. There is no restriction on p except for (4.1.1), and all other points have the same restrictions as in the proof of Lemma 4.1.22. Hence the maximum weight is the maximum weight of $M(K_{2,b})$ plus the extra weight from p , giving $2b + 3$. \square

Lemma 4.1.29. *Let N be a component of $\mathcal{L}(M, e)$ such that $N \cong M(K_{2,1}^+)$. Then*

- (i) *if $w_i = 5$ for some $i \in E(N)$, then $w_j = 3$ for all $j \in E(N) \setminus \{i\}$, and*
- (ii) *if $w_i = 4$ for some $i \in E(N)$, then $w_j = 3$ for some $j \in E(N) \setminus \{i\}$.*

Hence the maximum value of $w(N)$ is five.

Proof. Note that $M(K_{2,1}^+) \cong U_{2,3}$. Let $E(N) = \{x, y, z\}$ such that $w_x \geq w_y \geq w_z$.

First, assume that $w_x = 5$ and $w_y > 3$. Then, in M , there are three coplanar lines containing e ; one of length five, one of length at least four, and one of length at least three. However, this plane is not golden-mean by Lemma 2.3.1, and so $w_y = 3$, and case (i) follows.

Now assume that $w_z = 4$. If case (ii) fails, then, in M , there are three coplanar four-point lines containing e , meaning that M has a ten-element rank-three restriction with three copunctual lines, each of which contains at least four elements. This cannot happen by Lemma 2.3.1, and hence $w_z \neq 4$, and case (ii) follows.

It both cases, we can easily verify that $w(N) \leq 5$. □

Corollary 4.1.30. *The minimum regular deficit of $M(K_{2,b}^+)$ for some $b \geq 1$ is b .*

Corollary 4.1.31. *The minimum near-regular deficit of $M(K_{2,b}^+)$ for some $b \geq 1$ is -1 .*

Lemma 4.1.32. *Let N be a component of $\mathcal{L}(M, e)$ such that $N \cong U_{2,4}$. Then $w_i = 3$ for all $i \in E(N)$, and hence the maximum value of $w(N)$ is four.*

Proof. Assume that one point of $U_{2,4}$ corresponds to a line of length four. Then, in M , we have a ten-element rank-three restriction of M with all elements being on four lines, each of which contains e , leading to a contradiction to Lemma 2.3.1. Hence the maximum weight of $U_{2,4}$ is four. □

Corollary 4.1.33. *The minimum near-regular deficit of $U_{2,4}$ is zero.*

The only component of $\mathcal{L}(M, e)$ we have yet to consider is $U_{1,1}$. The only restriction on the element u of $U_{1,1}$ is (4.1.1). Hence $w_u \leq 5$, and so the maximum weight of $U_{1,1}$ is three.

Corollary 4.1.34. *The minimum regular deficit of $U_{1,1}$ is zero.*

Corollary 4.1.35. *The minimum near-regular deficit of $U_{1,1}$ is -1 .*

4.2 Lifts of Regular

We first prove Theorem 4.1.1. To do this, we consider a counterexample M of minimum rank to Theorem 4.1.1 and show that M does not exist. We are able to assume that M is maximum-sized. Note that as Lemma 2.3.1 characterises all the golden-mean matroids of rank three, we are able to assume that $r(M) \geq 4$.

First, we need to find an element e of M such that M/e is regular.

Lemma 4.2.1. *Let M be a maximum-sized member of \mathcal{R} . Then there exists $e \in E(M)$ such that M/e is regular.*

Proof. Assume for a counterexample that no such e exists. Then M has a single element extension N by f , where N is golden-mean and N/f is regular. However, M is maximum-sized, so f must be parallel to an element e of M . But then M/e is regular, a contradiction.

Hence such an e must exist. □

We now characterise $\mathcal{L}(M, e)$.

Lemma 4.2.2. *Let M be a counterexample to Theorem 4.1.1 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is regular. Then $\mathcal{L}(M, e)$ is a collection of coloops.*

Proof. As $\mathcal{L}(M, e)$ is a restriction of M/e , it follows that $\mathcal{L}(M, e)$ is regular. Hence we can apply Function 4.1.20 to get (4.1.5), which tells us that the sum of the regular deficits of all components of $\mathcal{L}(M, e)$ must be non-positive. From Corollaries 4.1.23, 4.1.26, 4.1.30, and 4.1.34, it is easy to see that the only way to make (4.1.5) true is if every component of $\mathcal{L}(M, e)$ is isomorphic to $U_{1,1}$. □

Corollary 4.2.3. *Let M be a rank- r counterexample to Theorem 4.1.1 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is regular. Then there are exactly $r - 1$ long lines through e , all of length five.*

Proof. If there are r long lines through e , then, as each long line corresponds to a coloop in $\mathcal{L}(M, e)$, the rank of $\mathcal{L}(M, e)$ must be at least r , and therefore $r(M/e) \geq r$, which is a contradiction.

Hence there can be at most $r - 1$ long lines containing e , and since $w(N) \leq 3$ for each $N \triangleleft \mathcal{L}(M, e)$, it follows that

$$\sum_{N \triangleleft \mathcal{L}(M, e)} w(N) + 1 \leq 3r - 2. \quad (4.2.1)$$

When we combine (4.1.3) with (4.2.1), we see that

$$\sum_{N \triangleleft \mathcal{L}(M, e)} w(N) + 1 = 3r - 2. \quad (4.2.2)$$

The only way to make (4.2.2) true is to have $r(\mathcal{L}(M, e)) = r - 1$ and $w(N) = 3$ for all N . Therefore there are $r - 1$ long lines containing e , all of length five. \square

Lemma 4.2.4. *Let M be a rank- r counterexample to Theorem 4.1.1 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is regular. Then $\text{si}(M/e) \cong M(K_r)$.*

Proof. As M is a counterexample to Theorem 4.1.1,

$$\epsilon(M) \geq \binom{r+3}{2} - 5.$$

It follows from Corollary 4.2.3 that

$$\epsilon(M) - \epsilon(M/e) = 3r - 2.$$

So

$$\epsilon(M/e) \geq \binom{r+3}{2} - 5 - 3r + 2 = \binom{r}{2}.$$

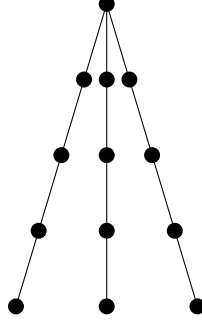


Figure 4.4: Forbidden Configuration from Lemma 4.2.5

As M/e is regular of rank at most $r - 1$, it follows from Theorem 1.1.9 that

$$\epsilon(M/e) \leq \binom{r}{2}.$$

Hence

$$\epsilon(M/e) = \binom{r}{2}$$

and it follows from Theorem 1.1.9 that $\text{si}(M/e) \cong M(K_r)$. \square

Lemma 4.2.5. *Let M be a rank- r counterexample to Theorem 4.1.1 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is regular. Then any two elements in $\mathcal{L}(M, e)$ will be on a triangle in $\text{si}(M/e)$.*

Proof. Firstly, note that it follows from Lemma 4.2.2 that the elements of $\mathcal{L}(M, e)$ form a basis of $\text{si}(M/e)$. Now let x be an element from $\text{si}(M/e)$ that is not in $\mathcal{L}(M, e)$. Assume that x is not in a line of $\text{si}(M/e)$ with two elements from $\mathcal{L}(M, e)$. If L_i and L_j are any long lines containing e , then $r(L_i \cup L_j) = 3$, and $r(L_i \cup L_j \cup x) = 4$. Hence $r_{M/x}(L_i \cup L_j) = 3$, so L_1, \dots, L_{r-1} are distinct lines of M/x . Hence $\mathcal{L}(M/x, e)$ has $r - 1$ points and rank at most $r - 2$, so $\mathcal{L}(M/x, e)$ contains a circuit, C .

If $|C| = 3$, then there exist three copunctal five-point lines in rank-three (Figure 4.4) in M/x , a contradiction to Lemma 2.3.1. If $|C| = 4$, then using the same technique as in Lemma 4.1.15 a 4-spike restriction of M/x can be found. However, as e is on five-point lines, the closures of the legs of this spike have cardinality five, leading to a contradiction to Corollary 4.1.7. Finally,

if $|C| \geq 5$ then a 5-spike can be found, contradicting Lemma 4.1.5. Hence C does not exist, so x must be on a triangle with two elements from $\mathcal{L}(M, e)$.

There are $\binom{r-1}{2}$ pairs of elements from $\mathcal{L}(M, e)$, and each of the $\binom{r}{2} - (r-1) = \binom{r-1}{2}$ elements from $\text{si}(M/e) \setminus \mathcal{L}(M, e)$ must be on a line with two elements from $\mathcal{L}(M, e)$. As $\text{si}(M/e)$ is binary, we have that every pair of elements from $\mathcal{L}(M, e)$ is on a triangle in $\text{si}(M/e)$. \square

Now we can prove Theorem 4.1.1. Let M be a rank- r counterexample to Theorem 4.1.1 of minimum rank. We can assume that M is maximum-sized. By Lemma 4.2.1, we can find $e \in E(M)$ such that M/e is regular.

We are going to construct a $GF(4)$ -representation for M . By Corollary 4.2.3, there are exactly $r - 1$ long lines passing through e , each of length five. Lemma 4.2.4 tells us that $\text{si}(M/e) \cong M(K_r)$.

Let L_i and L_j be lines passing through e . Then there is a unique element $f \in \text{si}(M/e)$ that is on the line in $\text{si}(M/e)$ between the point corresponding to $L_i - e$ and the point corresponding to $L_j - e$. Let us consider M restricted to the union of L_i , L_j , and f . This has rank three, and contains two lines of length five. When we contract f , we must get a copy of $U_{2,5}$ since M is representable over $GF(4)$. This means that f is on four different triangles. So the rank-three restriction is just a copy of T_3^2 , as in Figure 2.2.

Now pick an arbitrary element $x_i \in L_i$. For every other line L_j consider the element $f_{i,j}$ that is on the line between $L_i - e$ and $L_j - e$ in $\text{si}(M/e)$. Let x_j be the element of L_j that is contained in a triangle with x_i and $f_{i,j}$.

Assume that M is represented over $GF(4)$ by $[I_r|A]$, where the first column is labelled by e . Since $\text{si}(M/e)$ is isomorphic to $M(K_r)$ by Lemma 4.2.4, the uniqueness of representation of binary matroids over any field ([10, Proposition 6.6.5]) tells us that we can assume that

$$[I_r|A] = \left[I_r \left| \begin{array}{c} * \cdots * \\ D_{r-1} \end{array} \right| A' \right]$$

where the columns of I_r are labelled by (e, x_1, \dots, x_{r-1}) , and the columns of A' are the elements that are in five-point lines with e , but not in $\text{si}(M/e)$.

Since each x_i is in a five-point line with e , and e is on no other long lines, we

see that M must be represented by the matrix

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 0 \cdots 0 & 1 \cdots 1 & \alpha \cdots \alpha & \alpha^2 \cdots \alpha^2 & * \cdots * \\ \hline 0 & & & & & \\ \vdots & I_{r-1} & I_{r-1} & I_{r-1} & I_{r-1} & D_{r-1} \\ \hline 0 & & & & & \end{array} \right].$$

Each element in $\text{si}(M/e) \setminus \mathcal{L}(M, e)$, and hence each element represented by a column of the D_{r-1} block is in a triangle with two elements x_i and x_j , for some $\{i, j\} \subseteq \{1, \dots, r-1\}$, by Lemma 4.2.5. Therefore the row above the D_{r-1} block must contain zeros, and so M must be represented by the matrix

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 0 \cdots 0 & 1 \cdots 1 & \alpha \cdots \alpha & \alpha^2 \cdots \alpha^2 & 0 \cdots 0 \\ \hline 0 & & & & & \\ \vdots & I_{r-1} & I_{r-1} & I_{r-1} & I_{r-1} & D_{r-1} \\ \hline 0 & & & & & \end{array} \right].$$

This matrix is identical to the $GF(4)$ matrix for T_r^2 given in Figure 2.5.1, and hence $M \cong T_r^2$. However, this means that M is not a counterexample. Hence Theorem 4.1.1 is true.

From Theorem 4.1.1 we get the following two corollaries, which will be used in later work. The first corollary is easy, while the second needs a little work.

Corollary 4.2.6. *Let M be a counterexample to Conjecture 1.2.3. Then M has no element e such that M/e is regular.*

Corollary 4.2.7. *Let M be a counterexample to Conjecture 1.2.3 with no BR -minor. Then M has no copunctual five-point lines.*

Proof. Assume that M does have a point e such that e is on at least two five-point lines. By Corollary 4.2.6, M/e is not regular. Hence M/e is not binary, and therefore it has a $U_{2,4}$ -minor. Let x and y be elements corresponding to two five-point lines in $\mathcal{L}(M, e)$. By Theorem 3.2.4, M is vertically 4-connected, so $\text{si}(M/e)$ is 3-connected by Lemma 3.2.3. Now we can apply Lemma 2.4.2 and find a $U_{2,4}$ -minor using x and y in M/e . Hence when we de-contract e , in M , we can find a rank-three eleven-element restriction of M containing two copunctual five-point lines, which is a contradiction to Lemma 2.3.1.

Therefore there can be at most one five-point line going through e . \square

4.3 Lifts of Near-regular

We now prove Theorem 4.1.2 by showing that no counterexamples exist. We can assume that a counterexample is maximum-sized and of smallest possible rank. Note that as Lemma 2.3.1 characterises all the golden-mean matroids of rank-three, we are able to assume that $r(M) \geq 4$.

Throughout this section, we use the following results.

Lemma 4.3.1 (Bixby, Theorem 1 [2]). *Let M be a 3-connected matroid on E , and let $a \in E$. Then either $\text{co}(M \setminus a)$ or $\text{si}(M/a)$ is 3-connected.*

Lemma 4.3.2. *Let M be a 3-connected matroid, and let X be a non-empty subset of $E(M)$ such that X is a union of long lines from M . Then M has a 3-connected minor M' such that $M'|X = M|X$ and X spans M' .*

Proof. Assume that M is a minimal counterexample to the lemma. If X is the union of only one line, then $M' = M|X$ is the desired minor. Therefore X contains at least two lines. Let F be the closure of X in M . If $F = E(M)$ then we are done. Therefore we assume that there is a x in $E(M) \setminus E(F)$. By Lemma 4.3.1, either $\text{si}(M/x)$ or $\text{co}(M \setminus x)$ is 3-connected. Assume that the former holds. Let $N = \text{si}(M/x)$. We can assume that X is a subset of $E(N)$. Now $N|X = M|X$, and the minimality of M is contradicted. Therefore $\text{co}(M \setminus x)$ is 3-connected.

Let S contain all but one element from each non-trivial series class in $M \setminus x$. Thus $M \setminus x / S$ is isomorphic to $\text{co}(M \setminus x)$. Any circuit contained in S must intersect a series pair in a single element, which is impossible. Therefore S is independent in $M \setminus x$. The same argument shows that S is a flat, unless some series class of $M \setminus x$ is a circuit. But $M \setminus x$ is connected by [10, Proposition 8.2.8], so this would imply that $M \setminus x$ is a circuit and that M is a corank-2 uniform matroid. In this case M contains no lines, unless M is isomorphic to $U_{2,4}$, but this contradicts the fact that X does not span M . Therefore S is an independent flat in $M \setminus x$.

We will show that we can assume that S and F have an element in common. Assume otherwise, so that $S \cap F = \emptyset$. If $r(S) + r(F) = r(S \cup F)$, then

$(M \setminus x / S) \upharpoonright F = M \upharpoonright F$, and the minimality of M is contradicted. Therefore there is some circuit C contained in $S \cup F$ that contains elements from both S and F . Let S' be a series class in $M \setminus x$ such that S' contains elements of C . Then $S' \subseteq C$, for otherwise C intersects a series pair in a single element. The single element s in $S' \setminus S$ must therefore belong to F . If s is not in X , then there is a circuit C' contained in $X \cup \{s\}$ that contains s . In this case C' intersects a series pair in a single element. Therefore s is in X . Now we can use the set $(S \setminus \{s'\}) \cup \{s\}$ instead of S , where s' is an element of $S' \setminus \{s\}$. Therefore we can assume that s is an element in $S \cap X$.

Since every element in X is in a long line in $M \setminus x$, we can find a long line of $M \setminus x$ that intersects a series class. In order to avoid having a circuit meet a series pair in a single element, there must be a series class of size two in $M \setminus x$ that spans a circuit of size three. This implies that $\text{co}(M \setminus x)$ contains a parallel pair. Since $\text{co}(M \setminus x)$ is 3-connected, we see that $\text{co}(M \setminus x)$ is a restriction of $U_{1,3}$.

Switching to the dual, M^* / x is obtained from $U_{1,1}$, $U_{1,2}$, or $U_{2,3}$ by adding parallel elements. In the first and second cases M^* is a rank-2 uniform matroid. This means M must be isomorphic to $U_{2,4}$, and X must be the entire ground set, a contradiction. Therefore M^* consists of three lines in rank three, each of which contains x . Since X contains at least two lines in M , it follows that M^* contains at least two triads. This means that the lines containing x can contain no more than three elements. In fact, exactly one of them contains only two elements, so $|E(M)| = 6$. Thus $r(M) = r(M^*) = 3$, so M is spanned by X and we have a contradiction. \square

First, we need to find an element e of M such that M/e is near-regular.

Lemma 4.3.3. *Let M be a maximum-sized member of \mathcal{N} . Then there exists $e \in E(M)$ such that M/e is near-regular.*

Proof. Assume for a counterexample that no such e exists. Then M has a single-element extension N by f , where N is golden-mean and N/f is near-regular. However, M is maximum-sized, so f must be parallel to an element e of M . But then M/e is near-regular, a contradiction.

Hence such an e must exist. \square

Now let M be a maximum-sized counterexample to Theorem 4.1.2 of minimum rank. We will show that there are only two possible configurations for $\mathcal{L}(M, e)$. We start by restricting the lines through e in M .

Corollary 4.3.4. *Let M be a counterexample to Theorem 4.1.2. Then M has no copunctual five-point lines.*

Proof. If M has no BR -minor, then we can apply Corollary 4.2.7 to acquire the desired result. It is now sufficient to show that $BR \notin \mathcal{N}$. To this end, we use the Sage code `BR = matroids.named_matroids.BetsyRoss(); all([M.has_line_minor(5) for M in [BR.contract(e) for e in BR.groundset()]])`, which outputs `True`, telling us that BR/e contains a $U_{2,5}$ -minor, and hence is not near-regular, for all $e \in E(BR)$. Since BR is maximum-sized by Lemma 2.3.1 it follows that any golden-mean single-element extension of BR is a parallel extension. Therefore this check suffices to show that BR is not in \mathcal{N} . \square

Lemma 4.3.5. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If there exists an element f in $\mathcal{L}(M, e)$ such that $|\text{cl}_M(\{e, f\})| \geq 4$, then there can be no $U_{2,4}$ components in $\mathcal{L}(M, e)$.*

Proof. Assume that there is a $U_{2,4}$ component U of $\mathcal{L}(M, e)$. Lemma 4.1.32 states that $w_u = 3$ for all $u \in U$. Let

$$Z = \bigcup_{u \in U} \text{cl}_M(\{e, u\}) \text{ and } F = \text{cl}_M(\{e, f\}).$$

As every element of $U \cup \{f\}$ is from $\mathcal{L}(M, e)$, it follows that $Z \cup F$ is a union of long lines from M . Note that Z has rank three and consists of four copunctual three-point lines, and $F \notin \text{cl}_M(Z)$.

Now, by Lemma 4.3.2, we can find a 3-connected minor M' of M , such that $M'| (Z \cup F) = M|(Z \cup F)$ and $Z \cup F$ spans M' . As M' is 3-connected, there must be an element $g \in E(M')$ such that $g \notin \text{cl}_{M'}(Z) \cup \text{cl}_{M'}(F)$.

Then M'/g contains a rank-three restriction containing four copunctual long lines, one of which is very long. Hence M'/g is not golden-mean by Lemma 2.3.1, a contradiction. \square

Lemma 4.3.6. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If there exists an element f in $\mathcal{L}(M, e)$ with $|\text{cl}_M(\{e, f\})| \geq 4$, then there can be no $M(K_4)$ components in $\mathcal{L}(M, e)$.*

Proof. Assume that there is a $M(K_4)$ component K of $\mathcal{L}(M, e)$. Then Lemma 4.1.25 tells us that $w_k = 3$ for all $k \in E(K)$. Let

$$Z = \bigcup_{k \in K} \text{cl}_M(\{e, k\}) \text{ and } F = \text{cl}_M(\{e, f\}).$$

As every element of $K \cup \{f\}$ is from $\mathcal{L}(M, e)$, it follows that $Z \cup F$ is a union of long lines from M . Note that Z has rank four and consists of six copunctual three-point lines, and $F \notin \text{cl}_M(Z)$.

Now, by Lemma 4.3.2, we can find a 3-connected minor M' of M , such that $M'|(Z \cup F) = M|(Z \cup F)$ and $Z \cup F$ spans M' . As M' is 3-connected, there must be an element $g \in E(M')$ such that $g \notin \text{cl}_{M'}(Z) \cup \text{cl}_{M'}(F)$.

Now $M'/g/e$ contains a restriction L spanned by K such that $f \in \text{cl}_{M'/g/e}(K)$. There are three possible locations for f in L , relative to K :

- (i) Parallel to an element of K .
- (ii) Placed such that $\text{si}(M'/g/e) \cong F_7^-$.
- (iii) Placed such that $\text{si}(M'/g/e) \cong O_7$.

Hence we can find a four-element circuit C of $(M'/g)|L$, such that $f \in C$.

Let I and J be disjoint subsets of $E(M)$ such that $M \setminus I/J$ is equal to $(M'/g/e)|C$. Let

$$S = \bigcup_{c \in C} \text{cl}_M(\{e, c\}).$$

Now $M \setminus (I - S)/(J - S)$ is a rank-four matroid containing four copunctual long lines. The rank of any three of these lines is four, but at least one of them contains four elements, contradicting Corollary 4.1.7. \square

We can now show that $\mathcal{L}(M, e)$ contains no $U_{2,4}$ or $M(K_4)$ components.

Corollary 4.3.7. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then there are no components in $\mathcal{L}(M, e)$ isomorphic to either $U_{2,4}$ or $M(K_4)$.*

Proof. If $\mathcal{L}(M, e)$ contains such a component, then by Lemmas 4.3.5 and 4.3.6, $w_i = 3$ for all $i \in E(\mathcal{L}(M, e))$. Now it is easy to verify that the near-regular deficit of any component of $\mathcal{L}(M, e)$ is non-negative. This leads to a contradiction to (4.1.9). \square

We wish to restrict the values of w_i in various cases.

Lemma 4.3.8. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Assume that $\mathcal{L}(M, e)$ has distinct components N_1 and N_2 , both of which are isomorphic to $U_{2,3}$. Then there exists a four-element subset X of $E(N_1) \cup E(N_2)$ with the property that $w_x = 3$ for all $x \in X$.*

Proof. Assume for a contradiction that the lemma is false. Then there exists $Y \subset E(N_1) \cup E(N_2)$ such that $|Y| = 3$ and $w_y > 3$ for all $y \in Y$. Let

$$Z_i = \bigcup_{n \in E(N_i)} \text{cl}_M(\{e, n\}) \text{ for } i \in \{1, 2\}.$$

As every element of $N_1 \cup N_2$ is in $\mathcal{L}(M, e)$, it follows that $Z_1 \cup Z_2$ is a union of long lines of M .

By Lemma 4.3.2, there is a 3-connected rank-five minor M' of M such that $M|(Z_1 \cup Z_2) = M'|(Z_1 \cup Z_2)$. Since M' is 3-connected, there is an element g of M' that is in neither $\text{cl}_{M'}(Z_1)$ nor $\text{cl}_{M'}(Z_2)$.

Hence, in $M'/g/e$, the lines N_1 and N_2 are coplanar. This means that we can find a four-element circuit C of $M'/g/e$ such that $C \subseteq E(N_1) \cup E(N_2)$, and $C \cap Y \neq \emptyset$. Let I and J be disjoint subsets of $E(M)$ such that $M \setminus I/J$ is $(M'/g/e)|C$. Let

$$S = \bigcup_{c \in C} \text{cl}_M(\{e, c\}).$$

Now $M \setminus (I - S)/(J - S)$ is a rank-four matroid containing four copunctual long lines. The rank of any three of these lines is four, but at least one of them contains four elements, contradicting Corollary 4.1.7. \square

Lemma 4.3.9. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then there is at most one component N of $\mathcal{L}(M, e)$ such that $d_n(N) < 0$.*

Proof. Assume that the lemma fails. By Corollaries 4.1.24, 4.1.27, 4.1.31, 4.1.33, and 4.1.35, the minimum near-regular deficit of any component of $\mathcal{L}(M, e)$ is -1 . It follows that there are components N_1 and N_2 of $\mathcal{L}(M, e)$ such that $d_n(N_1) = d_n(N_2) = -1$. By Corollaries 4.1.24, 4.1.27, 4.1.31, 4.1.33, and 4.1.35, N_1 and N_2 are isomorphic to either $U_{1,1}$, $U_{2,3}$, or $M(K_{2,b}^+)$ for some $b \geq 2$.

Let $\{p, q\} = \{1, 2\}$. Assume N_p is isomorphic to $U_{2,3}$ and that $w_i < 5$ for every element i of N_p . By referring to Lemma 4.1.29, and the fact that $d_n(N_p) = -1$, we can assume that $w_x = w_y = 4$ and $w_z = 3$, where $\{x, y, z\}$ is the ground set of N_p .

Assume that there is an element j in N_q such that $w_j = 5$. Let

$$Z = \bigcup_{n \in N_p} \text{cl}_M(\{e, n\}) \text{ and } F = \text{cl}_M(\{e, j\}).$$

As every element of $N_p \cup \{j\}$ is from $\mathcal{L}(M, e)$, it follows that $Z \cup F$ is a union of long lines from M . Note that Z has rank three and consists of three copunctual long lines, two of which have length four, and $F \notin \text{cl}_M(Z)$.

Now, by Lemma 4.3.2, we can find a 3-connected minor M' of M , such that $M'|(Z \cup F) = M|(Z \cup F)$ and $Z \cup F$ spans M' . As M' is 3-connected, there must be an element $g \in E(M')$ such that $g \notin \text{cl}_{M'}(Z) \cup \text{cl}_{M'}(F)$.

Then M'/g contains a rank-three restriction containing three copunctual long lines, one of length five, and one of length four. Hence M'/g is not golden-mean by Lemma 2.3.1, a contradiction, so no such j can exist.

By considering Lemma 4.1.28 and the discussion prior to Corollary 4.1.34, we see that if N_q is isomorphic to either $U_{1,1}$ or $M(K_{2,b}^+)$ for some $b \geq 2$, then we have an element $j \in E(N_q)$ such that $w_j = 5$, which we have just shown is impossible. Hence $N_q \cong U_{2,3}$, and Lemma 4.1.29 tells us that the three elements of N_q have weights four, four, and three, respectively. This leads to a contradiction to Lemma 4.3.8.

Therefore neither N_p nor N_q is isomorphic to $U_{2,3}$, with all elements having weight strictly less than five. Now, as $d_n(N_1) = d_n(N_2) = -1$, we observe from Lemmas 4.1.28 and 4.1.29, and the discussion prior to Corollary 4.1.34, that there are points $a_i \in E(N_i)$ for $i \in \{1, 2\}$ such that $w_{a_i} = 5$. This contradicts Corollary 4.3.4, and we have completed the proof of the lemma. \square

Corollary 4.3.10. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then*

$$\sum_{N \triangleleft \mathcal{L}(M,e)} d_n(N) = -1.$$

Proof. By (4.1.9), the sum is at most -1 . By Corollaries 4.1.24, 4.1.27, 4.1.31, 4.1.33, and 4.1.35, the minimum near-regular deficit of any component of $\mathcal{L}(M,e)$ is -1 . Now the result follows from Lemma 4.3.9. \square

Lemma 4.3.11. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M,e)$ spans M/e , and $\text{si}(M/e)$ is isomorphic to T_{r-1}^1 .*

Proof. First we prove that $\mathcal{L}(M,e)$ spans M/e , that is, $r' = r(\mathcal{L}(M,e)) = r - 1$. Note that

$$r' \leq r - 1. \quad (4.3.1)$$

From Corollary 4.3.10 and the definition of Function 4.1.21 we can see that

$$\begin{aligned} -1 &= \sum_{N \triangleleft \mathcal{L}(M,e)} d_n(N) \\ &= \sum_{N \triangleleft \mathcal{L}(M,e)} 2r(N) - \sum_{N \triangleleft \mathcal{L}(M,e)} w(N) \\ &= 2r' - \sum_{N \triangleleft \mathcal{L}(M,e)} w(N). \end{aligned} \quad (4.3.2)$$

When we combine (4.3.2) with (4.1.7) and (4.3.1), we deduce

$$2r - 1 \leq \sum_{N \triangleleft \mathcal{L}(M,e)} w(N) = 2r' + 1 \leq 2(r - 1) + 1,$$

and thus equality holds throughout, and ergo $r' = r - 1$, as desired.

By considering (4.1.6), (4.1.8), and (4.3.2), along with the fact that $r' = r - 1$, we obtain

$$2r \leq \epsilon(M) - \epsilon(M/e) = \sum_{N \triangleleft \mathcal{L}(M,e)} w(N) + 1$$

$$= 2r' + 2 = 2(r - 1) + 2 = 2r. \quad (4.3.3)$$

As M/e is near-regular, Theorem 1.1.12 tells us that $\epsilon(M/e) \leq \binom{r+1}{2} - 2$. Also, as M is a counterexample to Theorem 4.1.2, $\epsilon(M) \geq \binom{r+3}{2} - 5$. Now (4.3.3) tells us that we must have equality in both of these expressions, so $\text{si}(M/e)$ is a maximum-sized near-regular matroid, and the result follows from Theorem 1.1.12. \square

We now prove some facts about restrictions of T_n^1 . We first need to define some terminology related to T_n^k for $k \in \{1, 2\}$.

Definition 4.3.12. Let k be 1 or 2. The element of T_n^k that is on $n - 1$ lines of length $k + 3$ is known as the **acme** of T_n^k . Each $(k + 3)$ -point line in T_n^k is known as a **leg**. A **contour** of T_n^k is a set of $n - 1$ points (one from each leg), such that every pair of elements are on a line with a non-leg element.

Lemma 4.3.13. Assume M is isomorphic to T_r^1 , for some $r \geq 3$. Let B be a restriction of M that is isomorphic to $M(K_{2,b}^+)$ where $b \geq 1$, and let p be the head of B (recall that if $b = 1$, then p is an arbitrary element of B). Let x be an element of M that is not in the span of B . Then there is a minor M' of M such that

- (i) $M'|B = M|B$,
- (ii) B spans M' ,
- (iii) $E(M') = E(B) \cup \{x\}$, and
- (iv) $r_{M'}(\{x, p\}) = 2$.

Proof. We prove this by induction on r . If $r = 3$, then $b = 1$. Up to symmetry there are three triangles of M that could be B :

1. a triangle that contains the acme of M ,
2. a triangle that spans a leg of M without containing the acme, and
3. the closure of a contour.

These are shown in Figure 4.5.

In all three cases, there are four or five points not in the span of B . By inspection, we see that by contracting points not in the span of B , any point not in the span of B can be projected into at least two different locations in the span of B . This establishes the lemma when $r = 3$.

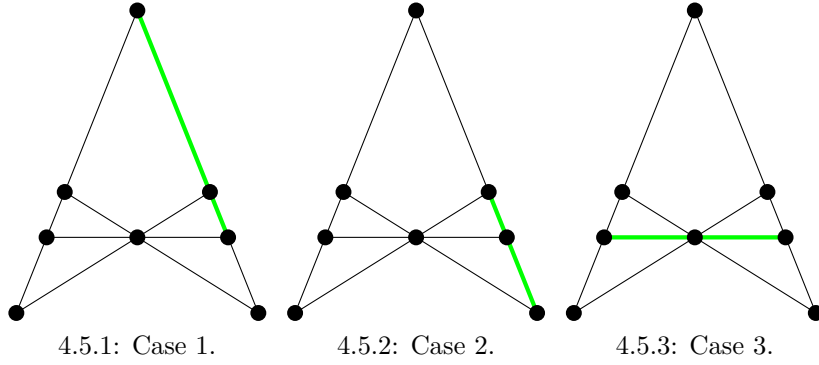
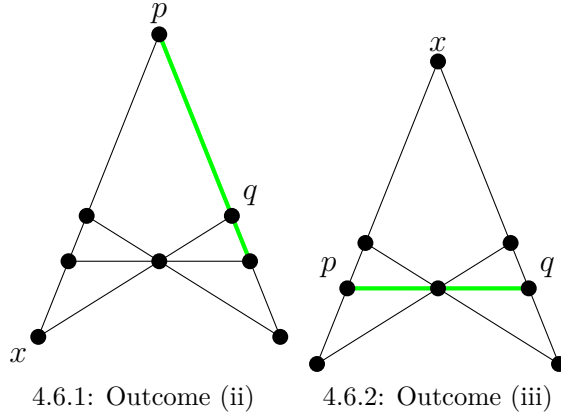
Figure 4.5: Three triangles of T_3^1 for Lemma 4.3.13

Figure 4.6: Outcomes (ii) and (iii) of Lemma 4.3.14

Now assume that $r > 3$ and that the lemma holds for $r - 1$. Since B is not spanning in M , we can find a cocircuit C^* such that $C^* \cap \text{cl}_M(B) = \emptyset$. In particular, we can assume that $x \in C^*$. Let z be the acme of M , and let $P = \text{cl}_M(\{p, x, z\})$. If C^* is a subset of P , then M contains a vertical 3-separation. However, this contradicts the fact that T_n^1 is vertically 4-connected [11, Lemma 5.1].

Therefore, we can let y be an element of $C^* \setminus P$. Now $\text{si}(M/y) \cong T_{r-1}^1$, since $y \neq z$. Since y is not an element of $\text{cl}_M(B)$, it follows that $\text{si}(M/y)|B = M|B$, so we can assume that $B \cup \{x\}$ is a subset of the ground set of $\text{si}(M/y)$. If x is in the closure of B in $\text{si}(M/y)$, then the restriction of $\text{si}(M/y)$ to $B \cup \{x\}$ is the desired minor, as x and p are not parallel in $\text{si}(M/y)$. On the other hand, if x is not in the closure of B , then we can apply the inductive hypothesis, and deduce that M' exists in any case. \square

Lemma 4.3.14. *Assume M is isomorphic to T_r^1 , for some $r \geq 3$. Let B be a $U_{2,3}$ -restriction of M containing distinct elements p and q . Let x be an element of $E(M)$ that is not in the span of B . Then there is a minor M' of M such that either*

- (i) $E(M') = B \cup \{x\}$, such that B is a spanning circuit of M' , and x is parallel to neither p or q , or
- (ii) M' is isomorphic to T_3^1 such that B is a circuit of M' with either p or q being the acme of M' , and x is the unique element of M' such that $\text{cl}_{M'}(\{x, p\})$ and $\text{cl}_{M'}(\{x, q\})$ are distinct long lines of M' , or
- (iii) M' is isomorphic to T_3^1 such that B is the closure of a contour of M' , and x is the acme of M' , while p and q are the points of B that form a contour of M' .

Up to symmetry, outcomes (ii) and (iii) are shown in Figure 4.6.

Proof. Let M be a minimum-rank counterexample to the lemma, such that $r(M) = r$. If $r > 3$, then there is a leg ℓ of M that is not contained in $\text{cl}_M(B \cup \{x\})$. Therefore there is a point y of ℓ that is not in $\text{cl}_M(B \cup \{x\})$ and that is not the acme of M . Hence $\text{si}(M/y) \cong T_{r-1}^1$, and we can assume that $B \cup \{y\}$ is a subset of $E(\text{si}(M/y))$. Therefore $\text{si}(M/y)$ is a lower rank counterexample to the lemma, which is not possible. Ergo $r = 3$.

Assume that B is the closure of a contour of M , as shown in Figure 4.5.3. If x is not the acme of M , then we let y be an element of $E(M) \setminus B$ such that y is not on a long line with x , and y is not the acme of M . In this case, $B \cup \{x\}$ is a $U_{2,4}$ -restriction in M/y , so outcome (i) holds. If x is the acme and outcome (i) does not hold, then $\{p, q\}$ must be the intersection of B and the legs of M , so outcome (iii) holds.

Now assume that B spans a leg in M , but that B does not contain the acme of M , as in Figure 4.5.2. Thus x is not the acme of M . If x is a non-leg element of M , then x can be projected so that it becomes parallel with any of the three elements of B , and therefore outcome (i) occurs. If x is a leg element of M , then it can be projected so that it becomes parallel with the acme, and hence outcome (i) holds again.

Finally, we assume that B contains the acme, as shown in Figure 4.5.1. If x is not a leg element of M , then we can easily verify that outcome (i) occurs.

Therefore x is a leg element of M , and the two points that are in $\text{cl}_M(B)$ and also on long lines with x must be p and q , and ergo outcome (ii) occurs. \square

We are now able to show that there are exactly two configurations of $\mathcal{L}(M, e)$.

Lemma 4.3.15. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M, e)$ is either*

- (i) *a collection of coloops, or*
- (ii) *isomorphic to $M(K_{2,r-2}^+)$.*

Proof. By Lemma 4.3.9 and Corollary 4.3.10, there is a single component of $\mathcal{L}(M, e)$ with near-regular deficit -1 , and all other components must have a near-regular deficit of zero. Let N_1, \dots, N_t be the components of $\mathcal{L}(M, e)$ where $d_n(N_1) = -1$. From Corollaries 4.1.24, 4.1.27, 4.1.31, 4.1.33, 4.1.35, and 4.3.7, we see that each N_i is isomorphic to $U_{1,1}$, $U_{2,3}$, or $M(K_{2,b}^+)$ where $b \geq 2$.

Assume N_1 is isomorphic to $U_{2,3}$ and $w_n < 5$ for all $n \in E(N_1)$. Therefore, by Lemma 4.1.29, we can assume that $w_x = w_y = 4$ and $w_z = 3$, where $\{x, y, z\}$ is the ground set of N_1 . Since $r > 3$, the rank of $\mathcal{L}(M, e)$ is at least three by Lemma 4.3.11, so $t > 1$. Since $d_n(N_2) = 0$, it follows from Lemmas 4.1.28 and 4.1.29, and the discussion prior to Corollary 4.1.34, that there is no element in N_2 with weight five, but there must be an element a_2 with weight four. We can assume that a_2 , x , y , and z are elements of $E(\text{si}(M/e))$, and we apply Lemma 4.3.14 to this matroid. It follows that M has a minor N such that either N is isomorphic to $U_{2,3}$ and contains x , y , and a_2 , or N is isomorphic to T_3^1 , where x , y , z , and a_2 are as in one of the configurations detailed by Lemma 4.3.14 and shown in Figure 4.6.

Let I and J be disjoint subsets of M such that $M \setminus I/J$ is N . Let

$$Z = \bigcup_{n \in E(N)} \text{cl}_M(\{e, n\}),$$

and consider $M \setminus (I - Z)/(J - Z)$. If N is isomorphic to $U_{2,3}$, then $M \setminus (I - Z)/(J - Z)$ is a rank-three matroid containing three copunctual very long lines, and hence is not golden-mean by Lemma 2.3.1. In the other two cases, $M \setminus (I - Z)/(J - Z)$ must be discovered by a computer search, using the Sage

code from Appendix 4.A.3. However, this computer search reports that no such golden-mean matroids exist.

Now we see that N_1 is isomorphic to either $U_{1,1}$ or $M(K_{2,b}^+)$ where $b \geq 1$. In the latter case if $b \geq 2$, then, by Lemma 4.1.28, the head of $M(K_{2,b}^+)$ is weighted five and all other elements of $M(K_{2,b}^+)$ are weighted three. If $b = 1$, then, by the previous paragraphs and Lemma 4.1.29, exactly one element, known as the head of $M(K_{2,1}^+)$, is weighted five, and the other two elements of $M(K_{2,1}^+)$ are weighted three.

If the statement of the lemma is false, then one of the following situations occurs:

- (i) N_1 is isomorphic to $U_{1,1}$, and, without loss of generality, N_2 is isomorphic to $M(K_{2,b}^+)$ for some $b \geq 1$.
- (ii) N_1 is isomorphic to $M(K_{2,b}^+)$ for some $b \geq 1$, and, without loss of generality, N_2 is isomorphic to $M(K_{2,d}^+)$ for some $d \geq 1$.
- (iii) N_1 is isomorphic to $M(K_{2,b}^+)$ for some $b \geq 1$, and, without loss of generality, N_2 is isomorphic to $U_{1,1}$.

We go through each case in turn and show that they are contradictory.

Sublemma 4.3.15.1. *Case (i) is contradictory.*

Subproof. Let X be a subset of $E(M/e)$ such that $M/e \setminus X \cong \text{si}(M/e)$, and let $M_e = M/e \setminus X$. By Lemma 4.3.11, we know that $M_e \cong T_{r-1}^1$. By the definition of case (i), we know that M_e has a $M(K_{2,b}^+)$ restriction B for some $b \geq 1$. Furthermore, $w(B) = 0$, and so Lemmas 4.1.28 and 4.1.29 tell us that the weight of every non-head element in B is three, while the head, p , of B is weighted four.

Now, there must be an element x of M_e such that $|\text{cl}_M(\{e, x\})| = 5$. Hence we can apply Lemma 4.3.13 to find a minor M' of M_e such that B spans M' and $E(M') = E(B) \cup \{x\}$, where x is not in parallel with p . We can now find a rank-two minor M'' of M' with at least three elements such that $r_{M''}(\{x, p\}) = 2$.

Consider the matroid M_Z spanned by Z , where

$$Z = \bigcup_{z \in E(M'')} \text{cl}_M(\{e, z\}).$$

Now, M_Z is a rank-three minor of M containing at least three long lines containing e , one of length five, and one of length four. This leads to a contradiction to Lemma 2.3.1, and so case (i) is contradictory, as required. \square

Sublemma 4.3.15.2. *Case (ii) is contradictory.*

Subproof. By previous work, we know that p , the head of N_1 , is weighted five. Consider the restriction of $\mathcal{L}(M, e)$ where N_1 is restricted to p , and apply Sublemma 4.3.15.1. Hence case (ii) is contradictory. \square

Sublemma 4.3.15.3. *Case (iii) is contradictory.*

Subproof. This is very similar to the proof of Sublemma 4.3.15.1. \square

As all three cases are contradictory, the lemma must be true. \square

From Lemma 4.3.15, we can deduce the values of $|\text{cl}_M(\{e, x\})|$ for all $x \in E(\mathcal{L}(M, e))$. The next two corollaries make this explicit.

Corollary 4.3.16. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is a collection of coloops, then there is one coloop y such that $|\text{cl}_M(\{e, y\})| = 5$, and $|\text{cl}_M(\{e, x\})| = 4$ for all $x \in E(\mathcal{L}(M, e)) \setminus \{y\}$.*

Corollary 4.3.17. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is isomorphic to $M(K_{2,b}^+)$, then $|\text{cl}_M(\{e, p\})| = 5$ where p is the head of $M(K_{2,b}^+)$, and $|\text{cl}_M(\{e, x\})| = 3$ for all $x \in E(\mathcal{L}(M, e)) \setminus \{p\}$.*

Lemma 4.3.11 tells us that $\text{si}(M/e) \cong T_{r-1}^1$, and Lemma 4.3.15 states that there are two possible solutions for $\mathcal{L}(M, e)$. Therefore we split the remainder of the proof of Theorem 4.1.2 into two natural cases.

4.3.1 All coloops

The first case to consider is when $\mathcal{L}(M, e)$ is a collection of coloops. In this case, Corollary 4.3.16 tells us exactly what the long lines containing e are.

Lemma 4.3.18. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is a collection of coloops, then $\mathcal{L}(M, e)$ is a basis of M/e .*

Proof. $\mathcal{L}(M, e)$ and M/e have the same rank. The result follows as $\mathcal{L}(M, e)$ is a collection of coloops. \square

We now show that $\mathcal{L}(M, e)$ must be a specific basis.

Lemma 4.3.19. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is a collection of coloops, then the basis of $\text{si}(M/e)$ corresponding to $\mathcal{L}(M, e)$ can have no fundamental circuit of size greater than three.*

Proof. Let B be this basis of $\text{si}(M/e)$, and let x be an element of $\text{si}(M/e)$ that is not in B . If $C \subseteq B \cup \{x\}$ is a fundamental circuit such that $|C| \geq 4$, then when we contract x and all but three points in C from $\text{si}(M/e)/x$, we get a three-point line, such that each point was from B . This leads to a rank-three restriction of M , with three copunctual four-point lines, a contradiction to Lemma 2.3.1. \square

Lemma 4.3.20. *The only basis of T_n^1 with all fundamental circuits having size three is the basis consisting of the acme of T_n^1 and a contour.*

Proof. We prove this by induction.

The lemma is true by inspection when $n = 3$, so now assume $n > 3$. Let B be a basis of T_n^1 . Assume that B contains no leg element, apart from possibly the acme. If e is an arbitrary leg element, then there is a line spanned by two elements of B that contains e . This line cannot contain the acme, so it consists of e , a non-leg element, and another leg element. Now we have a contradiction to the assumption that B contains no leg elements.

Hence we may assume that e in B is a leg element. In $\text{si}(M/e)$, the fundamental circuits of $B \setminus \{e\}$ have size at most three, and $\text{si}(M/e)$ is isomorphic to T_{n-1}^1 , so by induction, $B \setminus \{e\}$ consists of the acme, and a contour. If we pick f in this contour, then $B \setminus \{f\}$ consists of the acme and a contour, and it follows without difficulty that B consists of the acme and a contour of T_n^1 . \square

Corollary 4.3.21. *Let M be counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is a collection of coloops, then $\mathcal{L}(M, e)$ is the acme of $\text{si}(M/e)$ and a contour.*

Proof. This is an amalgamation of Lemmas 4.3.18, 4.3.19, and 4.3.20. \square

Corollary 4.3.16 states that there is one distinguished coloop in $\mathcal{L}(M, e)$, namely the coloop x such that $|\text{cl}_M(\{e, x\})| = 5$. We now show that x must correspond to the acme of $\text{si}(M/e)$.

Lemma 4.3.22. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. If $\mathcal{L}(M, e)$ is a collection of coloops, then the coloop $x \in \mathcal{L}(M, e)$ such that $|\text{cl}_M(\{e, x\})| = 5$ corresponds to the acme of $\text{si}(M/e)$.*

Proof. Lemma 4.3.11 tells us that $\text{si}(M/e) \cong T_{r-1}^1$, so the acme of $\text{si}(M/e)$ exists. Furthermore, Corollary 4.3.21 tells us that $\mathcal{L}(M, e)$ is an acme and a contour of $\text{si}(M/e)$.

Let ℓ_i be $\text{cl}_M(\{e, i\})$ for $i \in E(\mathcal{L}(M, e))$, and let a be the acme of $\text{si}(M/e)$.

Assume that $x \neq a$. Then $\text{cl}_M(\{e, a, x\})$ is a rank-three flat F_{ax} of M with ten elements, containing a four-point line (ℓ_a) and a five-point line (ℓ_x) that meet at e . Lemma 2.3.1 implies that $F_{ax} \cong G_3$.

Let N be a simple minor of M containing F_{ax} such that $N|_{F_{ax}} = M|_{F_{ax}}$ and $|\text{cl}_N(\ell_a)| = 5$. This means that $\text{cl}_N(\{e, a, x\})$ is an eleven-point rank-three flat with a five-point line restriction. This is a contradiction to Lemma 2.3.1, so such an N cannot exist.

Let y be an element of $E(\mathcal{L}(M, e)) \setminus \{a, x\}$. Then $\text{cl}_M(\{e, a, y\})$ is a nine-point rank-three flat F_{ay} of M containing ℓ_a . By analysis of the matroids listed in Lemma 2.3.1, we see that, in this case, we can contract an element

from $F_{ay} \setminus \ell_a$ and project a fifth point onto ℓ_a . Since $(F_{ay} \setminus \ell_a) \cap F_{ax} = \emptyset$, we have produced a minor N of the type described in the previous paragraph, a contradiction.

Hence $x = a$, as desired. \square

We now have enough information to prove that Conjecture 1.2.3 is true in this case.

Lemma 4.3.23. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M, e)$ cannot be a collection of coloops.*

Proof. We show that if $\mathcal{L}(M, e)$ is a collection of coloops, then $M \cong G_r$.

We are going to construct a $GF(4)$ -representation for M . We know that e is on $r-1$ long lines, one of length five, and $r-2$ of length four. Let $\ell_1, \dots, \ell_{r-1}$ be these long lines, where ℓ_1 has length five.

From Lemma 4.3.11, we know that $\text{si}(M/e) \cong T_{r-1}^1$. Let f_i be the element of $\mathcal{L}(M, e)$ that is also in ℓ_i , for $i \in \{1, \dots, r-1\}$.

As we have uniqueness of representation over $GF(4)$ by Lemma 2.1.12, we can assume that M is represented by a matrix of the form

$$\left[\begin{array}{c|c|c} I_r & A & \begin{array}{c} * \cdots * \\ T_{r-1} \end{array} \end{array} \right]$$

where the first r columns are labelled by (e, f_1, \dots, f_{r-1}) , and T_{r-1} is the matrix

$$\left[\begin{array}{c|c|c} 1 \cdots 1 & \alpha \cdots \alpha & 0 \cdots 0 \\ \hline I_{r-2} & I_{r-2} & D_{r-2} \end{array} \right].$$

The labelling is correct as Corollary 4.3.21 and Lemma 4.3.22 tell us exactly where $\mathcal{L}(M, e)$ is located in $\text{si}(M/e)$.

Since f_1 is on a five-point line with e , and e is on a four-point line with all other f_i s, we get that M must be represented by a matrix shown in Figure 4.7.1, where each \sharp is an element of $GF(4)$ and each $*$ is a non-zero element of $GF(4)$.

$$\begin{bmatrix}
 1 & 0 & 0 & \cdots & 0 & \# & \cdots & \# & \cdots & \# & \cdots & \# \\
 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\
 0 & 0 & & & & & & & & & & & & \\
 \vdots & \vdots & & & & & & & & & & & & \\
 0 & 0 & & & & & & & & & & & &
 \end{bmatrix}
 \begin{array}{c}
 1 \quad 1 \quad 1 \\
 1 \quad \alpha \quad \alpha^2 \\
 0_3^{r-2} \\
 I_{r-2} \\
 I_{r-2}
 \end{array}
 \begin{array}{c}
 * \quad \cdots \quad * \\
 0 \quad \cdots \quad 0 \\
 * \quad \cdots \quad * \\
 0 \quad \cdots \quad 0
 \end{array}$$

4.7.1: Constructing a matrix

$$\begin{bmatrix}
 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\
 0 & 0 & & & & & & & & & & & & \\
 \vdots & \vdots & & & & & & & & & & & & \\
 0 & 0 & & & & & & & & & & & &
 \end{bmatrix}
 \begin{array}{c}
 1 \quad 1 \quad 1 \\
 1 \quad \alpha \quad \alpha^2 \\
 0_3^{r-2} \\
 I_{r-2} \\
 I_{r-2}
 \end{array}
 \begin{array}{c}
 1 \quad \cdots \quad 1 \\
 0 \quad \cdots \quad 0 \\
 \alpha \quad \cdots \quad \alpha \\
 0 \quad \cdots \quad 0
 \end{array}$$

4.7.2: Final matrix

Figure 4.7: $GF(4)$ matrices for Lemma 4.3.23

Now consider the matroid $\text{si}(M/f_{r-1})$. Upon considering the matrix of M , we see that $\epsilon(M) - \epsilon(M/f_{r-1}) \leq r + 2$. However, we know that $\epsilon(M) - \epsilon(M/f_{r-1}) \geq r+2$, and hence $\epsilon(M) - \epsilon(M/f_{r-1}) = r+2$, so $\text{si}(M/f_{r-1}) \in \mathcal{G}_{r-1}$.

When we look at the matrix for $\text{si}(M/f_{r-1})$, we see a solitary five-point line and at least one four-point line. From the descriptions of the members of \mathcal{G}_{r-1} in Section 2.2.1, we note that G_{r-1} is the only member of \mathcal{G}_{r-1} having the required lines. Hence $\text{si}(M/f_{r-1}) \cong G_{r-1}$, and we are able to see that $\sharp = 0$. Also, we see that all except the last $*$ in each block are also fixed. To discover the last two values of $*$, we repeat this argument with f_{r-2} .

Hence M must be represented by the matrix in Figure 4.7.2, which is identical to the $GF(4)$ matrix for G_r given in Figure 2.5.2. Hence $M \cong G_r$.

Hence $\mathcal{L}(M, e)$ cannot be a collection of coloops. \square

4.3.2 $M(K_{2,r-2}^+)$

As $\mathcal{L}(M, e)$ cannot be a collection of coloops by Lemma 4.3.23, we can modify Lemma 4.3.15 to get that $\mathcal{L}(M, e) \cong M(K_{2,r-2}^+)$.

Corollary 4.3.24. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M, e)$ is isomorphic to $M(K_{2,r-2}^+)$.*

Recall from Corollary 4.3.17 that the head of $\mathcal{L}(M, e)$ corresponds to a five-point line of M , and every other point in $\mathcal{L}(M, e)$ corresponds to a three-point line of M .

We wish to know how $\mathcal{L}(M, e)$ is embedded inside of $\text{si}(M/e)$.

Lemma 4.3.25. *Let M be a counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then the head of $\mathcal{L}(M, e)$ must be identified with a leg element of $\text{si}(M/e)$.*

Proof. Let p be the head of $\mathcal{L}(M, e)$. For a contradiction, assume that p is a non-leg element of $\text{si}(M/e)$. Let I and J be disjoint subsets of $E(M/e)$ such that $M \setminus I/J = N$, where N is a T_3^1 restriction of $\text{si}(M/e)$ spanned by a $M(K_{2,2}^+)$ restriction K of $\mathcal{L}(M, e)$. Note that $p \in E(N)$.

Such an N exists as $\mathcal{L}(M, e)$ has $r - 2$ limbs, and a non-leg element of T_k^1 is on exactly $r - 2$ long lines, hence the T_3^1 -restriction of $\text{si}(M/e)$ that contains p must have two limbs of $\mathcal{L}(M, e)$ contained within it.

As p is a non-leg element of N , up to symmetry, there is only one place that K could go in N , namely consisting of two contours of N and p . Let f be a non-acme element of $E(N) \setminus E(K)$. Let

$$Z = \bigcup_{x \in E(\mathcal{L}(M, e))} \text{cl}_M(\{e, x\}).$$

Now consider the matroid $X = \text{si}(M \setminus (I - Z) / (J - Z) / f)$. In X , which has rank three, there are eleven points, five of which lie on a line. This is a contradiction to Lemma 2.3.1.

Hence the head of $\mathcal{L}(M, e)$ must be identified with a leg element of $\text{si}(M/e)$. \square

Lemma 4.3.26. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then the head of $\mathcal{L}(M, e)$ must be identified with the acme of $\text{si}(M/e)$.*

Proof. Let p be the head of $\mathcal{L}(M, e)$. From Lemma 4.3.25, we know that p must be identified with a leg element of $\text{si}(M/e)$. Assume for a contradiction that p is not the acme of $\text{si}(M/e)$.

As in the proof of Lemma 4.3.25, let N be a rank-three flat of $\text{si}(M/e)$ spanned by a $M(K_{2,2}^+)$ restriction K of $\mathcal{L}(M, e)$. Note that $p \in E(N)$, and that $N \cong T_3^1$.

Up to symmetry, there are two possible locations for K inside of N . One limb of K must be the closure of a contour containing p , and the other limb of K must be a subset S of the leg of N containing p .

If S contains the acme of N , then let f be the element of N that is

- (i) not in a contour with any element of K , and
- (ii) not on the same leg of N as p .

If S does not contain the acme of N , then let f be an element of $N \setminus K$ that is not the acme of N .

In both cases, when we consider $\text{si}(M/f)$, we can see an eleven-element rank-

three minor of M containing a five-point line, which is a contradiction to Lemma 2.3.1.

As all other possibilities have been rejected, the head of $\mathcal{L}(M, e)$ must be identified with the acme of $\text{si}(M/e)$. \square

Lemma 4.3.27. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then the non-head elements of $\mathcal{L}(M, e)$ must be identified with two contours of $\text{si}(M/e)$.*

Proof. Assume for a contradiction that this lemma is false. Then there exists a rank-three restriction N of $\text{si}(M/e)$ that is isomorphic to T_3^1 and has $\mathcal{L}(M, e)$ embedded in such a way that $K = \mathcal{L}(M, e)|N$ does not contain two contours of N .

There is one non-leg element f of N . When we consider $\text{si}(M/f)$, we can see an eleven-element rank-three minor of M containing a five-point line, which is a contradiction to Lemma 2.3.1.

Hence the non-head elements of $\mathcal{L}(M, e)$ must be identified with two contours in $\text{si}(M/e)$. \square

Recall the definition of T_r^1 from Page 60.

Note that in T_r^1 there are three contours: one belongs to the distinguished basis, and is known as the basis contour, and the other two (headed by 1 and α in the matrix of T_r^1) are known as non-basis contours.

The next two lemmas, whose proofs are similar to the proof of Lemma 4.3.23, show that Theorem 4.1.2 is true.

Lemma 4.3.28. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M, e)$ cannot contain the basis contour of $\text{si}(M/e)$.*

Proof. We show that if $\mathcal{L}(M, e)$ does contain the basis contour of $\text{si}(M/e)$, then $M \cong HP_r$. So assume for a contradiction that $\mathcal{L}(M, e)$ contains the basis contour of $\text{si}(M/e)$.

We are going to construct a $GF(4)$ -representation of M . We know that e is on $2r - 3$ long lines, one of length five and $2(r - 2)$ of length three. Let $\ell_1, \dots, \ell_{2r-3}$ be these long lines, where ℓ_1 has length five, and $\ell_2, \dots, \ell_{r-1}$ correspond to the basis contour of $\text{si}(M/e)$.

From Lemma 4.3.11, we know that $\text{si}(M/e) \cong T_{r-1}^1$. Let f_i be the element of $\mathcal{L}(M, e)$ that is also in ℓ_i , for $i \in \{1, \dots, 2r - 3\}$.

As we have uniqueness of representation over $GF(4)$ by Lemma 2.1.12, we can assume that M is represented by a matrix of the form

$$\left[\begin{array}{c|c|c} I_r & A & \begin{array}{c} * \cdots * \\ T_{r-1} \end{array} \end{array} \right]$$

where the first r columns are labelled by (e, f_1, \dots, f_{r-1}) , and T_{r-1} is the matrix

$$\left[\begin{array}{c|c|c} 1 \cdots 1 & \alpha \cdots \alpha & 0 \cdots 0 \\ \hline I_{r-2} & I_{r-2} & D_{r-2} \end{array} \right].$$

The labelling is correct as we are assuming that $\mathcal{L}(M, e)$ contains the acme and the basis contour of $\text{si}(M/e)$.

Since f_1 is on a five-point line with e , and e is on a three-point line with all other f_i s, we get that M must be represented by a matrix shown in Figure 4.8.1, where each \sharp is an element of $GF(4)$ and each $*$ is a non-zero element of $GF(4)$.

Now consider the matroid $\text{si}(M/f_{r-1})$. Upon considering the matrix of M , we see that $\epsilon(M) - \epsilon(M/f_{r-1}) \leq r + 2$. However, we know that $\epsilon(M) - \epsilon(M/f_{r-1}) \geq r + 2$, and hence $\epsilon(M) - \epsilon(M/f_{r-1}) = r + 2$, so $\text{si}(M/f_{r-1}) \in \mathcal{G}_{r-1}$.

When we look at the matrix for $\text{si}(M/f_{r-1})$, we see a solitary five-point line and no four-point lines. Hence $\text{si}(M/f_{r-1}) \cong HP_{r-1}$, and we are able to determine the values for \sharp . Also, we see that all except the last $*$ in each block are also fixed. To discover the last two values of $*$, we repeat this argument with f_{r-2} .

Hence M must be represented by the matrix in Figure 4.8.2, which is identical to the $GF(4)$ matrix for HP_r given in Figure 2.5.3. Hence $M \cong HP_r$.

4.8.1: Constructing a matrix

4.8.2: Final matrix

Figure 4.8: $GF(4)$ matrices for Lemma 4.3.28

Therefore $\mathcal{L}(M, e)$ cannot contain the basis contour of $\text{si}(M/e)$. \square

We now know exactly how $\mathcal{L}(M, e)$ is embedded in $\text{si}(M/e)$.

Corollary 4.3.29. *Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank, and assume that M is maximum-sized. Let $e \in E(M)$ be such that M/e is near-regular. Then $\mathcal{L}(M, e)$ is isomorphic to $M(K_{2,r-2}^+)$, with the head of $\mathcal{L}(M, e)$ identified with the acme of $\text{si}(M/e)$, and the limbs of $\mathcal{L}(M, e)$ correspond to the two non-basis contours of $\text{si}(M/e)$.*

We can now prove Theorem 4.1.2. Let M be a rank- r counterexample to Theorem 4.1.2 of minimum rank. We can assume that M is maximum-sized. By Lemma 4.3.3, we can find $e \in E(M)$ such that M/e is near-regular.

We are going to construct a $GF(4)$ representation for M . By Corollary 4.3.17, we know there are exactly $2r - 3$ long lines passing through e , one of length five, and $2(r - 2)$ of length three. Lemma 4.3.11 tells us that $\text{si}(M/e) \cong T_{r-1}^1$, and Corollary 4.3.29 tells us how $\mathcal{L}(M, e)$ is embedded in $\text{si}(M/e)$.

Let $\ell_1, \dots, \ell_{2r-3}$ be the long lines of M that contain e , where ℓ_1 is the line of length five. Let f_i be the element of $\mathcal{L}(M, e)$ that is also in ℓ_i , for $i \in \{1, \dots, 2r - 3\}$.

As we have uniqueness of representation over $GF(4)$ by Lemma 2.1.12, we can assume that M is represented by a matrix of the form

$$\left[\begin{array}{c|c|c} I_r & A & \begin{array}{c} * \cdots * \\ T_{r-1} \end{array} \end{array} \right]$$

where the first r columns are labelled by $(e, f_1, g_1, \dots, g_{r-2})$, and T_{r-1} is the matrix

$$\left[\begin{array}{c|c|c} 1 \cdots 1 & \alpha \cdots \alpha & 0 \cdots 0 \\ \hline & & \\ I_{r-2} & I_{r-2} & D_{r-2} \end{array} \right].$$

with the first $2(r - 2)$ columns labelled by f_2, \dots, f_{2r-3} .

The labelling is correct as we know that $\mathcal{L}(M, e)$ consists of the acme and the two non-basis contours of $\text{si}(M/e)$.

$$\begin{bmatrix}
 1 & 0 & 0 & \dots & 0 & \# & \dots & \# & \dots & \# & \dots & \# & \dots & * & \dots & * \\
 0 & 1 & 0 & \dots & 0 & 1 & \dots & 1 & \alpha & \dots & \alpha & 0 & \dots & 0 & \dots & \alpha \\
 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0
 \end{bmatrix}$$

4.9.1: Constructing a matrix

$$\begin{bmatrix}
 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 \\
 0 & 1 & 0 & \dots & 0 & 1 & \dots & 1 & \alpha & \dots & \alpha & 0 & \dots & 0 & \dots & 0 \\
 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0
 \end{bmatrix}$$

4.9.2: Final matrix

Figure 4.9: $GF(4)$ matrices for the proof of Theorem 4.1.2

Since f_1 is on a five-point line with e , and e is on a three-point line with all other f_i s, we get that M must be represented by a matrix shown in Figure 4.9.1, where each \sharp is an element of $GF(4)$ and each $*$ is a non-zero element of $GF(4)$.

Now consider the matroid $\text{si}(M/g_{r-2})$. Upon considering the matrix of M , we see that $\epsilon(M) - \epsilon(M/g_{r-1}) \leq r + 2$. However, we know that $\epsilon(M) - \epsilon(M/g_{r-1}) \geq r + 2$, and hence $\epsilon(M) - \epsilon(M/g_{r-1}) = r + 2$, so $\text{si}(M/g_{r-1}) \in \mathcal{G}_{r-1}$.

When we look at the matrix for $\text{si}(M/g_{r-1})$, we see a solitary five-point line and at least one four-point line. Hence $\text{si}(M/g_{r-1}) \cong G_{r-1}$, and we are able to determine the values for \sharp . Also, we see that all except the last $*$ in each block are also fixed. To discover the last two values of $*$, we repeat this argument with g_{r-2} .

Hence M must be represented by the matrix in Figure 4.9.2. When we take the $GF(4)$ matrix for G_r given in Figure 2.5.2 and add the first row to the second row, we get the matrix for M from Figure 4.9.2. Hence $M \cong G_r$.

However, this means that M is not a counterexample to Theorem 4.1.2. Hence Theorem 4.1.2 is true.

4.4 A Strategic Hurdle

The proof of Theorem 1.1.17, which characterises the maximum-sized sixth-roots-of-unity matroids, proceeds in much the same way that the results in this chapter go. Oxley, Vertigan, and Whittle [11] assume that M/e is regular for some e , where M is a maximum-sized sixth-roots-of-unity matroid, and e is an element of M . Once they eliminate this case, they consider the case where M/e is non-regular and use Lemma 2.4.2 to derive contradictions. These contradictions arise from placing a $U_{2,4}$ -minor on two points of $\mathcal{L}(M, e)$, and then lifting e . They then show that this lift cannot be sixth-roots-of-unity, a contradiction.

In our proof, this final case proves troublesome. We have a 2-rounded set, $\{U_{2,5}, U_{3,5}, F_7^-, (F_7^-)^*, Q_6\}$, by Lemma 2.4.3. However, it is possible to lift each member of this set while still remaining golden-mean. This was verified using the Sage code in Appendix 4.A.4. We then try to extend Defini-

Appendix 4.A Code

```

spikes4 = []
F = GF(4, 'x')
x = F.gens()[0]
poss = [0, x, x+1]
for a in poss:
    for b in poss:
        for c in poss:
            for d in poss:
                mat = matrix(GF(4, 'x'), [[1, a, 1, 1, 1], [1, 1, b, 1, 1],
                                             [1, 1, 1, c, 1], [1, 1, 1, 1, d]])
                M = Matroid(reduced_matrix=mat)
                if M.is_simple():
                    spikes4.append(M)
print [a, len(spikes4)]

print "isomorphism_now_4"
spikes4 = get_nonisomorphic_matroids(spikes4)

```

```
spikes5 = []
poss = [0, 2, 3, 4]
for a in poss:
    for b in poss:
        for c in poss:
            for d in poss:
                mat = matrix(GF(5), [[1, a, 1, 1, 1], [1, 1, b, 1, 1],
                    [1, 1, 1, c, 1], [1, 1, 1, 1, d]])
```

```

M = Matroid(reduced_matrix=mat)
if M.is_simple():
    spikes5.append(M)
print [a, len(spikes5)]

print "isomorphism_now_(5)"
spikes5 = get_nonisomorphic_matroids(spikes5)

final = matroid_set_intersection(spikes4, spikes5)

```

This code constructs a $GF(4)$ -representation of all golden-mean 4-spikes. We know what the matrix for a 4-spike must look like, and we construct all possible 4-spikes over both $GF(4)$ and $GF(5)$. We then call the function `matroid_set_intersection()` from Appendix 2.A.2 to get the $GF(4)$ 4-spikes that are isomorphic to a $GF(5)$ 4-spike, which results in the golden-mean 4-spikes by Theorem 1.2.2.

4.A.2 Lemma 4.1.5

```

spikes4 = []
F = GF(4, 'x')
x = F.gens()[0]
poss = [0, x, x+1]
for a in poss:
    for b in poss:
        for c in poss:
            for d in poss:
                for e in poss:
                    mat = matrix(GF(4, 'x'), [[1, a, 1, 1, 1, 1], [1, 1,
                        b, 1, 1, 1], [1, 1, 1, c, 1, 1], [1, 1, 1, 1, d,
                        1], [1, 1, 1, 1, 1, e]])
                    M = Matroid(reduced_matrix=mat)
                    if M.is_simple():
                        spikes4.append(M)
print [a, len(spikes4)]

print "isomorphism_now_(4)"

```

```

spikes4 = get_nonisomorphic_matroids(spikes4)

spikes5 = []
poss = [0, 2, 3, 4]
for a in poss:
    for b in poss:
        for c in poss:
            for d in poss:
                for e in poss:
                    mat = matrix(GF(5), [[1, a, 1, 1, 1, 1], [1, 1, b, 1,
                        1, 1], [1, 1, 1, c, 1, 1], [1, 1, 1, 1, d, 1],
                        [1, 1, 1, 1, 1, e]])
                    M = Matroid(reduced_matrix=mat)
                    if M.is_simple():
                        spikes5.append(M)
print [a, len(spikes5)]

print "isomorphism_now_(5)"
spikes5 = get_nonisomorphic_matroids(spikes5)

final = matroid_set_intersection(spikes4, spikes5)

```

This code is almost identical to the code in Appendix 4.A.1. The difference is that it constructs 5-spikes instead of 4-spikes.

4.A.3 Lemma 4.3.15

```

# Make T for comparison
TCC = {2: ['abdg', 'acfh', 'beh', 'def', 'ceg'], 3: ['abcdefgh']
    }
T = Matroid(groundset='abcdefgh', circuit_closures = TCC)

start = identity_matrix(GF(19), 3)
A = Matroid(matrix=start, groundset='abc')
poss = []

for B in A.linear_extensions(element='d', fundamentals=funds,

```

```

    simple=True, F='ab'):
for C in B.linear_extensions(element='e', fundamentals=funds,
    simple=True, F='ab'):
    for D in C.linear_extensions(element='f', fundamentals=
    funds, simple=True, F='ac'):
    for E in D.linear_extensions(element='g', fundamentals=
    funds, simple=True, F='ac'):
    for F in E.linear_extensions(element='h', fundamentals=
    funds, simple=True, F='eg'):
    if F.is_isomorphic(T):
        poss.append(F)

for H in poss:
    for I in H.linear_coextensions(element='i', fundamentals=
    funds):
    for J in I.linear_extensions(element='j', fundamentals=
    funds, simple=True, F='bi'):
    for K in J.linear_extensions(element='k', fundamentals=
    funds, simple=True, F='bi'):
    for L in K.linear_extensions(element='l', fundamentals=
    funds, simple=True, F='ai'):
    for M in L.linear_extensions(element='m',
    fundamentals=funds, simple=True, F='ai'):
    for N in M.linear_extensions(element='n',
    fundamentals=funds, simple=True, F='fi'):
    for O in N.linear_extensions(element='o',
    fundamentals=funds, simple=True, F='fi'):
    for P in O.linear_extensions(element='p',
    fundamentals=funds, simple=True, F='gi'):
        P

for H in poss:
    for I in H.linear_coextensions(element='i', fundamentals=
    funds):
    for J in I.linear_extensions(element='j', fundamentals=

```

```

    funds, simple=True, F='ei'):
    for K in J.linear_extensions(element='k', fundamentals=
        funds, simple=True, F='ei'):
        for L in K.linear_extensions(element='l', fundamentals=
            funds, simple=True, F='ai'):
            for M in L.linear_extensions(element='m',
                fundamentals=funds, simple=True, F='ai'):
                for N in M.linear_extensions(element='n',
                    fundamentals=funds, simple=True, F='gi'):
                    for O in N.linear_extensions(element='o',
                        fundamentals=funds, simple=True, F='gi'):
                        for P in O.linear_extensions(element='p',
                            fundamentals=funds, simple=True, F='hi'):
                            P

```

In the proof of Lemma 4.3.15, we come across a case where a counterexample to the lemma must have one of two very specific structures, detailed in cases (ii) and (iii) of Lemma 4.3.14. This code starts by constructing all possible golden-mean representations (over $GF(19)$) for T_3^1 , and stores these in `poss`. We then attempt to construct the counterexamples to Lemma 4.3.15, by creating a lift of T_3^1 , and extending into various lines. If any such matroids exist, we then print them. Upon running this code, nothing is output, so no counterexamples to Lemma 4.3.15 (in this particular case) exist.

4.A.4 Section 4.4

```

U24 = matroids.Uniform(2, 4)
U25 = matroids.Uniform(2, 5)
Q6 = matroids.named_matroids.Q6()
F7ECC = {2: ['abc', 'afe', 'adg', 'cdf', 'ceg'], 3: ['abcdefg'
    ]}
F7E = Matroid(groundset='abcdefg', circuit_closures = F7ECC)

# start with a U25
u25_reps = []
for vect in Tuples(range(19), 2):
    reduced_matrix = matrix(GF(19), [[1, 1, 1], [1, vect[0],

```

```

    vect[1]]])
M = Matroid(reduced_matrix=reduced_matrix, groundset='abcde
    ')
if M.is_isomorphic(U25):
    if is_gm(M):
        u25_reps.append(M)
        u25_reps.append(M.dual())
print "U25sumade"

# now make the Q6s
q6_reps = []
for 0 in u25_reps:
    for M in 0.linear_coextensions(element='f', fundamentals=
        funds):
        if M.is_isomorphic(Q6):
            q6_reps.append(M)
print "Q6uconstructed"

# and finally F7e
U24_reps = []
for vect in Tuples(list(funds), 1):
    reduced_matrix = matrix(GF(19), [[1, 1], [1, vect[0]]])
    M = Matroid(reduced_matrix=reduced_matrix, groundset='abcd'
        )
    if M.is_isomorphic(U24):
        if is_gm(M):
            U24_reps.append(M)
coextended = []
for 0 in U24_reps:
    for M in 0.linear_coextensions(element='e', fundamentals=
        funds):
        coextended.append(M)
print "coextended"
f7_reps = []
for M in coextended:

```

```

for N in M.linear_extensions(element='f', simple=True,
    fundamentals=funds, F='ae'):
    for 0 in N.linear_extensions(element='g', simple=True,
        fundamentals=funds, F='ad'):
        if 0.is_isomorphic(F7E):
            f7_reps.append(0)
            f7_reps.append(0.dual())

print "F7EUconstructed"

# now lift
lifted = []

for M in u25_reps:
    for N in M.linear_coextensions(element='f', fundamentals=
        funds):
        lifted.append(N)

print "liftedU(U25)"

for M in q6_reps:
    for N in M.linear_coextensions(element='g', fundamentals=
        funds):
        lifted.append(N)

print "liftedU(Q6)"

for M in f7_reps:
    for N in M.linear_coextensions(element='h', fundamentals=
        funds):
        lifted.append(N)

print "liftedU(F7=)"

baddies = []

for M in lifted:
    gs = M.groundset_list()
    long = len(gs)
    end = gs[long - 1]
    for xy in Combinations(range(long - 1), 2):
        for N in M.linear_extensions(simple=True, fundamentals=
            funds, F=[end, gs[xy[0]]]):

```

```

    for O in N.linear_extensions(simple=True,
                                fundamentals=funds, F=[end, gs[xy[0]]]):
        for P in O.linear_extensions(simple=True,
                                    fundamentals=funds, F=[end, gs[xy[1]]]):
            for Q in P.linear_extensions(simple=True,
                                        fundamentals=funds, F=[end, gs[xy[1]]]):
                for R in Q.linear_extensions(simple=
                    True, fundamentals=funds, F=[end, gs
                        [xy[1]]]):
                    baddies.append(R)
print "now_isomorphism_out"
baddies = get_nonisomorphic_matroids(baddies)

```

This code takes a matroid \mathbf{M} from our 2-rounded set $\{U_{2,5}, U_{3,5}, F_7^=, (F_7^=)^*, Q_6\}$, and coextends it freely by the element e (stored in `end`). We then choose two points x and y of \mathbf{M} (stored in `xy`), and extend the line $\{x, e\}$ to a four-point line and the line $\{y, e\}$ to a five-point line. This produces two matroids (up to isomorphism), which means that a contradiction is not forthcoming.

My Panasonic typewriter can make graphs. It types in four different colours.

Heather O'Rourke

5

\mathbb{G} -graphic Matroids

5.1 Preliminaries

Definition 5.1.1 (Van Zwam [20, Definitions 3.2.1 and 3.2.2]). A matroid M is **\mathbb{P} -graphic** for some partial field \mathbb{P} if there exists a \mathbb{P} -matrix A with at most two non-zero entries per column such that M is represented over \mathbb{P} by A .

In this chapter, we prove the following theorem.

Theorem 5.1.2. *Let \mathcal{T} be the class of golden-mean-graphic matroids. Then*

$$h_{\mathcal{T}}(r) = \binom{r+3}{2} - 5.$$

Furthermore, $M \in \mathcal{T}$ is maximum-sized if and only if M is isomorphic to either $T_{r(M)}^2$ or $G_{r(M)}$.

Let \mathfrak{M} be a \mathbb{G} -matrix representing a loopless matroid M with at most two non-zero entries per column. From \mathfrak{M} we are able to construct a directed

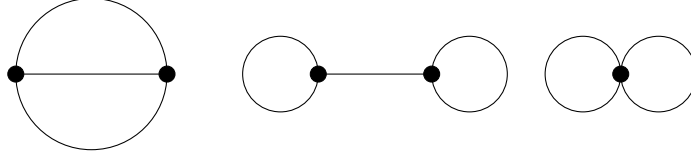


Figure 5.1: From left to right: the theta, barbell, and handcuff graphs

weighted graph W . Firstly, scale each column of \mathfrak{M} so that the first non-zero entry is one. Each row of \mathfrak{M} corresponds to a vertex, and each column corresponds to an edge. If a column of \mathfrak{M} contains non-zero entries in rows i and j , where row i is above row j , then that column corresponds to a directed edge from i to j , and the weight of this edge is the entry from row j . If a column of \mathfrak{M} contains only one non-zero entry, in row i , then that column corresponds to a loop on i , weighted one.

Thus each edge in W corresponds to an element of M . A cycle C in W is **balanced** if the product of its edge weights is equal to one. Let c be an edge from C , with weight w_c . If c is directed in accordance with the direction on C , then we use w_c in our product. If c is directed counter to the direction on C , then we use w_c^{-1} in our product.

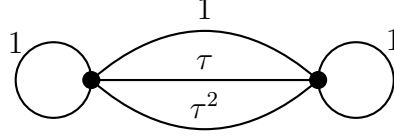
Lemma 5.1.3 (Zaslavsky [26, Matroid Theorem]). *Let M be a \mathbb{P} -graphic matroid with associated weighted directed graph W . Then the circuits of M correspond to balanced cycles of W ; or to theta, barbell, or handcuff graphs (see Figure 5.1) that contain no balanced cycles.*

In order to prove Theorem 5.1.2, we require \mathcal{T} to be minor-closed.

Lemma 5.1.4 (Van Zwam [20, Proposition 3.2.3]). *Let \mathbb{P} be a partial field. The class of \mathbb{P} -graphic matroids is minor-closed.*

Lemma 5.1.5 (Zaslavsky [27]). *Let M be a \mathbb{G} -graphic matroid, and let W be a weighted directed graph associated with M . Let W' be a minor of W . Then there is a minor M' of M such that M' has a \mathbb{G} -graphic representation corresponding to W' .*

Let M be a simple \mathbb{G} -graphic matroid, and let W be a corresponding weighted directed graph. If e is a non-loop edge in W , then W/e is the corresponding graph for the \mathbb{G} -graphic matroid M/e . On the other hand, if e is a loop of W

Figure 5.2: \mathbb{G} -graphic representation of $U_{2,5}$

at the vertex v , then we let G' be the graph obtained from G by deleting v , and replacing non-loop edges $u_i v$ incident with v by loops that are incident with u_i . Then G' corresponds to M/e , with loops deleted.

Example 5.1.6. The matroid $U_{2,5}$ has the \mathbb{G} -graphic representation shown in Figure 5.2, and is represented over \mathbb{G} by the following matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \tau & \tau^2 \end{bmatrix} \quad \diamond$$

5.1.1 Graphs With No K_4

Definition 5.1.7 (Diestel [3, Page 11]). Let G be a connected graph. If v is a vertex such that $G \setminus v$ is disconnected, then v is a **cut vertex**.

Definition 5.1.8 (Diestel [3, Page 60]). A **block** of G is a maximal connected subgraph without a cut vertex.

Lemma 5.1.9 (Oxley [10, Lemma 5.4.11]). *A simple 2-connected graph in which the degree of every vertex is at least three has a subgraph that is a subdivision of K_4 .*

The following result, which is related to series-parallel networks, is well-known. A proof is given here for completeness.

Lemma 5.1.10. *Let G be a simple graph with no K_4 -minor. Then G has a vertex of degree at most two.*

Proof. Assume for a contradiction that all vertices of G have degree at least three. Without loss of generality, we can assume that G is connected. Amongst connected counterexamples, assume that G has been chosen so that it has the smallest possible number of edges.

Assume that G is 2-connected. Then G is a simple 2-connected graph in which the degree of every vertex is at least three, so Lemma 5.1.9 applies, and we deduce that G has a K_4 -minor. However, this is a contradiction to the definition of G , so G cannot be 2-connected.

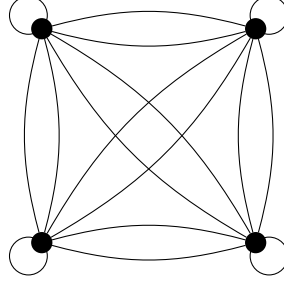
Hence G is connected but not 2-connected, and must have at least two blocks.

An end-block is a block of G that is incident with only one cut-vertex. It is well-known (see [3, Lemma 3.1.4] for example) that every graph that is connected but not 2-connected must have at least two end-blocks. Consider two of the end-blocks, B_1 and B_2 , of G . Let v_i be the cut vertex between B_i and $G \setminus B_i$, for $i \in \{1, 2\}$.

Since B_1 and B_2 have fewer edges than G , they each have at least one vertex of degree at most two. In fact, the only vertex in B_i of degree at most two is v_i , for otherwise, the vertex with degree at most two in B_i would also be in G . If v_i has degree one in B_i , then since B_i is a block, it follows that B_i contains a single edge, and hence G contains a vertex of degree one. Therefore v_i has degree exactly two in B_i .

Now let G' be the graph obtained from B_1 and B_2 by identifying them at the vertices v_1 and v_2 . Our previous paragraph shows that G' has no vertices of degree at most two. As G' is a minor of G , it must be equal to G , or else the minimality of G is contradicted. Let c be the unique cut-vertex in G . Note that c has degree four.

Let u_i and w_i be the two vertices incident with c in B_i , for $i \in \{1, 2\}$. Consider the graph G'' , where $V(G'') = V(G) \setminus \{c\}$ and $E(G'') = E(G) \cup \{u_1u_2, w_1w_2\}$. G'' cannot have a K_4 -minor as neither B_1 nor B_2 has a K_4 -minor, and, as K_4 has no 2-separation, any K_4 -minor of G'' would use at most one vertex of either B_1 or B_2 , which would imply that B_1 or B_2 has a K_4 -minor, a contradiction. As $|E(G)| - |E(G'')| = 2$, the minimality of G implies that G'' has a vertex of degree at most two. This vertex must have the same degree in G , hence G must have a vertex of degree at most two. \square

Figure 5.3: The graph $2K_4^\circ$

5.2 Results

Lemma 5.2.1. *Let M be a maximum-sized \mathbb{G} -graphic matroid. Let W be a weighted directed graph that corresponds to a representation of M . Then W has a loop at every vertex.*

Proof. Let \mathfrak{M} be a \mathbb{G} -matrix representing M such that each column of \mathfrak{M} has at most two non-zero entries and \mathfrak{M} corresponds to W . If W does not have a loop at every vertex, then \mathfrak{M} does not contain every standard basis vector. Adding standard basis vectors as columns to \mathfrak{M} does not change its sub-determinants, and thus we are able to do so, contradicting the fact that M is maximum-sized. \square

Definition 5.2.2. The graph $2K_4^\circ$ is the graph K_4 with every edge being a parallel class of size two, and a single loop at every vertex. This is shown in Figure 5.3.

Lemma 5.2.3. *Let W be a weighted directed graph corresponding to a \mathbb{G} -graphic matroid. Then W has no $2K_4^\circ$ -minor.*

Proof. We prove this using Sage [19]. The code is in Appendix 5.A.1. The algorithm we use constructs a representation for $2K_4^\circ$ over $GF(19)$ and fills it out using elements of the set $\{1, \dots, 19\}$, and then tests for \mathbb{G} -representability using the function `is_gm()` from Appendix 2.A.2. As no results are given, this shows that $2K_4^\circ$ is not a \mathbb{G} -graphic matroid, and hence W cannot have a $2K_4^\circ$ -minor by Lemma 5.1.5. \square

Lemma 5.2.4. *Let M be a non-ternary \mathbb{G} -graphic matroid, and let W be a weighted directed graph that corresponds to a representation of M . Let e and f be loops of W . Then M has a minor M' where*

- (i) W' is a weighted directed graph that corresponds to a representation of M' ,
- (ii) W' is a minor of W ,
- (iii) e and f are loops in W' , and
- (iv) M' is isomorphic to $U_{2,5}$.

Proof. As M is non-ternary, it must have one of the excluded minors for $GF(3)$ -representability (see [10, Theorem 10.2.1]) as a minor. Because neither F_7 nor F_7^* is $GF(5)$ -representable, we see that M must have either $U_{2,5}$ or $U_{3,5}$ as a minor.

Hence we can apply Lemma 2.4.3, so M has a minor N isomorphic to a member of $\{U_{2,5}, U_{3,5}, F_7^=, (F_7^=)^*, Q_6\}$ that uses e and f . Now conditions (i) and (ii) hold by Lemmas 5.1.4 and 5.1.5. Condition (iii) holds as loops of N , where $N \leq M$, are loops of M .

From Example 5.1.6, we know that it is possible for N to be isomorphic to $U_{2,5}$. In this case, we take M' to be N and we are done.

We now show that N cannot be isomorphic to $U_{3,5}$, $F_7^=$, or $(F_7^=)^*$.

Sublemma 5.2.4.1. *It is not possible for N to be isomorphic to $U_{3,5}$, $F_7^=$, or $(F_7^=)^*$.*

Subproof. We prove this by attempting to construct a $GF(4)$ -graphic representation for each matroid, and failing in each case. The code used is in Appendix 5.A.2. \square

Hence N must be isomorphic to Q_6 . We now show that we are able to find M' in this case.

Sublemma 5.2.4.2. *If N is isomorphic to Q_6 , then there is an element x of Q_6 , such that x does not correspond to a loop in the weighted graph that corresponds to N , and such that N/x is isomorphic to $U_{2,5}$.*

Subproof. We prove this by constructing all $GF(4)$ -graphic representations for Q_6 with at least two loops. We then contract a non-loop element, and discover a matroid isomorphic to $U_{2,5}$. The code is in Appendix 5.A.3. \square

Hence we can always find M' , as desired. \square

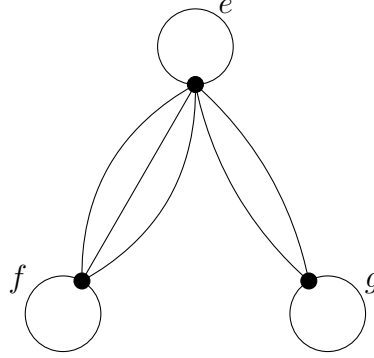


Figure 5.4: Labels for Lemma 5.2.5

Lemma 5.2.5. *Let M be a counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Then a four-point line cannot meet a five-point line.*

Proof. Assume for a contradiction that the lemma is false. In W , the weighted directed graph associated with M , the five-point line must correspond to two loops, and three edges joining them, and the four-point line must correspond to two loops with two edges joining them. If e is the element in both lines, then e is a loop. Let f and g be the other two loops, as shown in Figure 5.4.

Consider the matroid $\text{si}(M/e)$. We can assume that $\text{si}(M/e)$ is non-ternary as, if it were ternary, then we could apply Theorem 4.1.2. Let Z be a subset of $E(M)$ such that $M/e \setminus Z \cong \text{si}(M/e)$. We can assume that neither f nor g is in Z . Then, by Lemma 5.2.4, $M/e \setminus Z$ has a $U_{2,5}$ -minor N using f and g . Let I and J be disjoint subsets of $E(M/e \setminus Z)$ such that $M/e \setminus Z / I \setminus J = N$. Now, the matroid $M/I \setminus J$ is a rank-three matroid with eleven points, and hence is isomorphic to BR by Lemma 2.3.1. However, BR does not contain a five-point line, so we have arrived at a contradiction. \square

Lemma 5.2.6. *Let M be a rank- r counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Then there are no five-point lines in M .*

Proof. Assume there is a five-point line ℓ in M . The line ℓ must have two loops in W , the weighted directed graph associated with M , labelled f and g . Since $r(M) \geq 4$, there has to be a non-loop edge e incident with f or

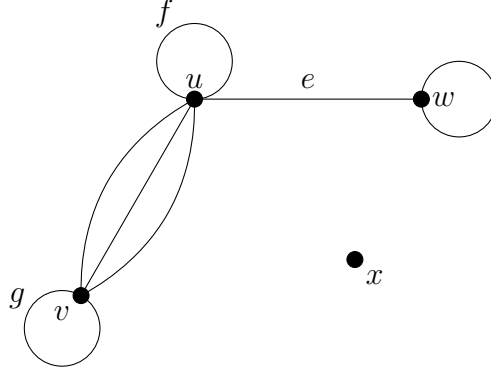


Figure 5.5: Labels for Lemma 5.2.6

g , or else W is not connected, implying that M is not connected. Without loss of generality, we can assume that e is incident with f . This is shown in Figure 5.5.

Now consider $\epsilon(M) - \epsilon(M/e)$. First, e is not in a four or five-point line, since that line would have to contain f , contradicting Lemma 5.2.5 or Corollary 4.2.7. There is one barbell (Figure 5.1) containing e , and every other three-point line containing e corresponds to a balanced triangle in W . We claim that there is at most one such triangle incident with every vertex other than the end-vertices of e . Using the labels from Figure 5.5, note that there is at most one edge between v and w , since otherwise g is in a four-point line because every vertex is incident with a loop by Lemma 5.2.1. Therefore we can see that e is in at most one balanced triangle using the vertex w . If x is a vertex of W that is not u , v , or w , then there is at most one edge between u and x , for the same reason, and therefore e is in at most one balanced triangle incident with x . Now e is in no $U_{2,4}$ -restriction, and in at most $r - 1$ three-point lines, so $\epsilon(M) - \epsilon(M/e)$ is at most r .

However, as M is a counterexample to Theorem 5.1.2, it must be that $\epsilon(M) \geq \binom{r+3}{2} - 5$. Furthermore, the minimality of M implies that $\epsilon(M/e) \leq \binom{r+2}{2} - 5$. Hence $\epsilon(M) - \epsilon(M/e)$ is at least $r + 2$.

We now have a contradiction, and hence there can be no five-point lines in M . \square

Let X be the subgraph induced by removing all loops and one edge from every parallel class in W . Observe that there is an edge between u and v in X if and only if there are multiple edges between u and v in W .

Corollary 5.2.7. *Let M be a counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Then X , as constructed above, has no K_4 -minor.*

Proof. If X has a K_4 -minor, then W , the weighted directed graph associated with M , has a $2K_4^\circ$ -minor, which contradicts Lemma 5.2.3. \square

Corollary 5.2.8. *Let M be a counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Then X , as constructed above, has a vertex of degree two.*

Proof. As Corollary 5.2.7 shows, X has no K_4 -minor, and hence Lemma 5.1.10 applies and so X has a vertex of degree two. \square

Lemma 5.2.9. *Let M be a rank- r counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Let W be the weighted directed graph associated with M , and let e be a loop in W that is incident with the vertex discovered in Corollary 5.2.8. Then $\text{si}(M/e)$ is isomorphic to either T_{r-1}^2 or G_{r-1} .*

Proof. Construct X as described above, and apply Corollary 5.2.8 to find e . Note that edges in X correspond to very long lines in M . As there are no five-point lines in M by Lemma 5.2.6, each edge in X corresponds to a four-point line in X . Hence e can be on at most two four-point lines in M . Therefore, e is on at most two four-point lines and $r - 3$ three-point lines. Let Z be a subset of $E(M)$ such that $M/e \setminus Z \cong \text{si}(M/e)$. Z can have at most one element from each three-point line and two elements from each four-point line. Thus the maximum value of $|Z|$ is $(r - 3) + (2 \times 2) = r + 1$. So

$$\epsilon(M) - \epsilon(M/e) = |Z| + 1 \leq r + 2. \quad (5.2.1)$$

Furthermore, since M is a minimal counterexample, we have that $\epsilon(M) \geq \binom{r+3}{2} - 5$, and that $\epsilon(M/e) \leq \binom{r+2}{3} - 5$. Hence

$$\epsilon(M) - \epsilon(M/e) \geq r + 2. \quad (5.2.2)$$

When we combine (5.2.1) with (5.2.2), we see that all inequalities in those two equations are equalities. Hence $\text{si}(M/e)$ is the same size as a maximum-

sized golden-mean matroid, and the result follows from the minimality of M . \square

We are now able to prove Theorem 5.1.2. Let M be a rank- r counterexample to Theorem 5.1.2 of minimum rank, and assume that M is maximum-sized. Let W be the weighted directed graph associated with M , and let e be the loop of W contracted in Lemma 5.2.9.

Consider $\text{si}(M/e)$. By Lemma 5.2.9, when we construct $\text{si}(M/e)$ we get either G_{r-1} or T_{r-1}^2 , both of which contain a five-point line. However, as e is a loop, it is not possible for $\text{si}(M/e)$ to have five-point lines that are not present in M . Hence M has a five-point line, which is a contradiction to Lemma 5.2.6.

Therefore there can be no counterexamples to Theorem 5.1.2, and so the theorem is true, as desired.

Appendix 5.A Code

5.A.1 Lemma 5.2.3

```
r = range(19)
C = Combinations(r, 2)

for gh in C:
    for ij in C:
        for mn in C:
            for f in r:
                for l in r:
                    for p in r:
                        tester = matrix(GF(19),
[[1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0],
[0, 1, 0, 0, 1, f, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1],
[0, 0, 1, 0, 0, 0, gh[0], gh[1], 1, 1, 0, 0, 0, 0, 0, 0],
1, p],
[0, 0, 0, 1, 0, 0, 0, 0, ij[0], ij[1], 1, l, mn[0],
mn[1], 0, 0]])
```

```

M = Matroid(tester)
if is_gm(M):
    print [f, gh, ij, l, mn, p]

```

This code, which takes a long time, tests the graph $2K_4^\circ$ to see if it could be a directed weighted graph corresponding to a \mathbb{G} -graphic matroid. As nothing is output, there is no \mathbb{G} -graphic matroid with corresponding directed weighted graph isomorphic to $2K_4^\circ$.

5.A.2 Sublemma 5.2.4.1

5.A.2.1 $U_{3,5}$

```

U35 = matroids.Uniform(3,5)
F = GF(4, 'x')
x = F.gens()[0]
std = [[1,0,0],[0,1,0],[0,0,1]]
vecs = []
for a in [1, x, x^2]:
    vecs.append([1,a,0])
    vecs.append([1,0,a])
    vecs.append([0,1,a])

poss = []
# 2 loops
for loops in Combinations(std, 2):
    for thing in Combinations(vecs, 3):
        A = matrix(F, [loops[0], loops[1], thing[0], thing[1],
            thing[2]]).transpose()
        M = Matroid(A)
        poss.append(M)

# 3 loops
for thing in Combinations(vecs, 2):
    A = matrix(F, [std[0], std[1], std[2], thing[0], thing[1]]).
        transpose()
    M = Matroid(A)

```

```
poss.append(M)
```

```
any([M.is_isomorphic(U35) for M in poss])
```

This code creates all simple rank-three five-element $GF(4)$ -graphic matroids with at least two loops and stores them in `poss`. It then checks to see if any members of `poss` are isomorphic to $U_{3,5}$, and outputs `False`, so no members of `poss` are isomorphic to $U_{3,5}$.

5.A.2.2 F_7^\perp

```
F7ECC = {2: ['abc', 'aef', 'cde', 'beg', 'cfg'], 3: ['abcdefg']
        ]}
F7E = Matroid(groundset='abcdefg', circuit_closures = F7ECC)
F = GF(4, 'x')
x = F.gens()[0]
std = [[1,0,0],[0,1,0],[0,0,1]]
vecs = []
for a in [1, x, x^2]:
    vecs.append([1,a,0])
    vecs.append([1,0,a])
    vecs.append([0,1,a])

poss = []
# 2 loops
for loops in Combinations(std, 2):
    for thing in Combinations(vecs, 5):
        A = matrix(F, [loops[0], loops[1], thing[0], thing[1],
                        thing[2], thing[3], thing[4]]).transpose()
        M = Matroid(A)
        poss.append(M)

# 3 loops
for thing in Combinations(vecs, 4):
    A = matrix(F, [std[0], std[1], std[2], thing[0], thing[1],
                    thing[2], thing[3]]).transpose()
    M = Matroid(A)
```

```
poss.append(M)
```

```
any([M.is_isomorphic(F7E) for M in poss])
```

This code creates all simple rank-three seven-element $GF(4)$ -graphic matroids with at least two loops and stores them in `poss`. It then checks to see if any members of `poss` are isomorphic to F_7^- , and outputs `False`, so no members of `poss` are isomorphic to F_7^- .

5.A.2.3 $(F_7^-)^*$

```
F7ECC = {2: ['abc', 'aef', 'cde', 'beg', 'cfg'], 3: ['abcdefg']}
    }
F7E = Matroid(groundset='abcdefg', circuit_closures = F7ECC)
F7D = F7E.dual()
F = GF(4, 'x')
x = F.gens()[0]
std = [[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]
vecs = []
for a in [1, x, x^2]:
    vecs.append([1,a,0,0])
    vecs.append([1,0,a,0])
    vecs.append([1,0,0,a])
    vecs.append([0,1,a,0])
    vecs.append([0,1,0,a])
    vecs.append([0,0,1,a])

poss = []
# 2 loops
for loops in Combinations(std, 2):
    for thing in Combinations(vecs, 5):
        A = matrix(F, [loops[0], loops[1], thing[0], thing[1],
            thing[2], thing[3], thing[4]]).transpose()
        M = Matroid(A)
        poss.append(M)

# 3 loops
```

```

for loops in Combinations(std, 3):
    for thing in Combinations(vecs, 4):
        A = matrix(F, [loops[0], loops[1], loops[2], thing[0],
            thing[1], thing[2], thing[3]]).transpose()
        M = Matroid(A)
        poss.append(M)

# 4 loops
for thing in Combinations(vecs, 4):
    A = matrix(F, [std[0], std[1], std[2], std[3], thing[0],
        thing[1], thing[2]]).transpose()
    M = Matroid(A)
    poss.append(M)

any([M.is_isomorphic(F7D) for M in poss])

```

This code creates all simple rank-four seven-element $GF(4)$ -graphic matroids with at least two loops and stores them in `poss`. It then checks to see if any members of `poss` are isomorphic to $(F_7^=)^*$, and outputs **False**, so no members of `poss` are isomorphic to $(F_7^=)^*$.

5.A.3 Sublemma 5.2.4.2

```

Q6 = matroids.named_matroids.Q6()
U25 = matroids.Uniform(2,5)
F = GF(4, 'x')
x = F.gens()[0]
std = [[1,0,0],[0,1,0],[0,0,1]]
vecs = []
for a in [1, x, x^2]:
    vecs.append([1,a,0])
    vecs.append([1,0,a])
    vecs.append([0,1,a])

# 2 loops
for loops in Combinations(std, 2):
    for thing in Combinations(vecs, 4):

```

```

A = matrix(F, [loops[0], loops[1], thing[0], thing[1],
               thing[2], thing[3]]).transpose()
M = Matroid(matrix=A, groundset=range(6))
if M.is_isomorphic(Q6):
    any([M.contract(x).is_isomorphic(U25) for x in
         [2,3,4,5]])

# 3 loops
for thing in Combinations(vecs, 3):
    A = matrix(F, [std[0], std[1], std[2], thing[0], thing[1],
                   thing[2]]).transpose()
    M = Matroid(matrix=A, groundset=range(6))
    if M.is_isomorphic(Q6):
        any([M.contract(x).is_isomorphic(U25) for x in [3,4,5]])

```

This code constructs all simple rank-three six-element $GF(4)$ -graphic matroids with at least two loops and checks each of them to see if they are isomorphic to Q_6 . If a representation is, we then contract each of the non-loop elements and check to see whether any of the resulting matroids are isomorphic to $U_{2,5}$.

...the handyman's secret weapon, duct tape.

Red Green

6

Matroids with a Spanning Clique

Warning: In this chapter, we make heavy use of colour.

6.1 Preliminaries

We start with some definitions that will be used throughout this chapter.

Definition 6.1.1. Let M be a rank- r matroid with a $M(K_{r+1})$ -restriction. We call this restriction a *spanning clique* for M .

Definition 6.1.2. Let M be a matroid with a spanning clique N . A *non-clique line* l is a line of M such that there exists an $e \in l$ such that $\{e\} \cap E(N) = \emptyset$.

Definition 6.1.3. Let G be a graph. A *star* in G is a $K_{1,n}$ subgraph of G , where $n \geq 2$. The vertex of the star that all edges of the star are incident with is called the *core* of the star.

In this chapter, we prove the following theorem. This proves Conjecture 1.2.3 in the class of golden-mean matroids with a spanning clique.

Theorem 6.1.4. *Let \mathcal{M} be the set of golden-mean matroids that have a spanning clique. Let \mathcal{C} be the family of minors of matroids in \mathcal{M} . Then*

$$h_{\mathcal{C}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{C}$ is maximum-sized if and only if M is isomorphic to a member of $\mathcal{G}_{r(M)}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

In Section 6.7, we outline how we anticipate that recent work by Geelen and Nelson (personal correspondence), when combined with Theorem 6.1.4, will lead to a proof of Conjecture 1.2.3 for matroids with sufficiently large rank.

In order to prove this theorem, we need to know that it is true for ranks three and four. Rank-three follows from Lemma 2.3.1. For rank four, we have the following lemma.

Lemma 6.1.5. *Theorem 6.1.4 is true when $r = 4$.*

Proof. We prove this with a computer search, the code for which is in Appendix 6.A.1. This also follows from independent work by Archer [1], Hliněný (personal correspondence), and Pendavingh (personal correspondence). \square

To prove Theorem 6.1.4, which we do in Section 6.6, we first show that there are only two ways to extend onto a spanning clique in a golden-mean way. We show that in any golden-mean matroid with a spanning clique there must be an element of the clique that is not on a $U_{2,4}$ -restriction. In order to prove this, we need to characterise all the ways that a clique element can be on a $U_{2,4}$ -restriction, which we do in Section 6.5. We then prove that if a clique element e in a counterexample to Theorem 6.1.4 is not on a $U_{2,4}$ -restriction, then e is on at least three non-clique three-point lines. In Section 6.4, we discover that there are four possible configurations that arise when e is on three non-clique three-point lines. We then show that it is not possible to make these configurations big enough to be counterexamples to Theorem 6.1.4, and hence no counterexamples can exist.

In the next definition, we introduce some useful informal language.

Definition 6.1.6. Let G be a graph, and let A and B be two subgraphs of G . We say that A *touches* B if there exist vertices v_A and v_B in A and B respectively such that $v_A = v_B$ in G .

6.2 Red-Green Lemma

There are two ways to extend onto a clique in a golden-mean way. For simplicity, these are called “green” (corresponding to identifying a $M(K_4)$ -restriction of the spanning clique and extending it to an O_7) and “red” (corresponding to identifying a $M(K_4)$ -restriction of the spanning clique and extending it to a F_7^-).

First, we need to show that this claim is actually true.

Lemma 6.2.1. *If M is a simple golden-mean matroid with a non-coloop element e such that $M \setminus e$ is isomorphic to $M(K_{r+1})$, then e is in a line of cardinality three or four.*

Proof. Note that if e is in a line, this line cannot have cardinality greater than four, as the maximum size of a line in a graphic matroid is three.

We prove the lemma by induction on r . Now consider a minimal counterexample M , so M is an extension by e of the graphic matroid $M(G)$, where G is a complete graph on $r + 1$ vertices, and e is not on any long lines in M . Let x be an arbitrary edge in G . Note that in M/x , the element e is not in a parallel pair. Let P contain all but one element from each parallel class in M/x . So $M/x \setminus P$ is an extension of $M(K_r)$ by e . Hence by induction, e is on a line with some elements y and z in $M/x \setminus P$. Therefore $e \in \text{cl}_M(\{x, y, z\})$.

Let V be the set of vertices of G incident with at least one of x , y , or z ; and let G' be the complete subgraph of G induced by V . If we consider $M' = M|(E(G') \cup \{e\})$, this is a counterexample, as e is a non-coloop element with $M' \setminus e$ being the cycle matroid of a complete graph, and e is not in any long lines in M' . Hence, by minimality, $G' = G$, and therefore $r + 1$ is at most 6.

We now have a finite case check, which can be done on a computer. When $r = 2$, the proof is trivial. So we only need to check values of r in $\{3, 4, 5\}$, which is done in [Appendix 6.A.2](#). \square

Lemma 6.2.2. *The line obtained in Lemma 6.2.1 is unique.*

Proof. Take a minimal counterexample M . So, in M , there exist distinct long lines ℓ_1 and ℓ_2 , such that $\ell_1 \cap \ell_2 = \{e\}$, and $|\ell_1| \geq |\ell_2|$.

If $|\ell_1| = 4$, then we have a 4-point line (with three clique elements) and a line with at least two clique elements (f and g) meeting at e . Note that M/f contains a four-point line consisting of ℓ_1 , with $\{e, g\}$ being a parallel pair. Deleting e from this gives us a graphic matroid with a $U_{2,4}$ -restriction, a contradiction [10, Theorem 6.6.7]. Therefore $|\ell_1| = |\ell_2| = 3$.

Say that $\ell_1 = \{e, x_1, y_1\}$ and $\ell_2 = \{e, x_2, y_2\}$. This means that $\{x_1, x_2, y_1, y_2\}$ is a four-element cycle in our underlying graph. For $i \in \{1, 2\}$, note that x_i and y_i are not in a triangle, as then ℓ_i would be a 4-point line. So there is a K_4 -subgraph of our underlying graph with two edge-disjoint matchings, such that e is on a line with each of them. This makes a Fano or non-Fano restriction of M , which is a contradiction as neither the Fano nor the non-Fano is golden-mean [10, Page 643]. \square

Because of Lemmas 6.2.1 and 6.2.2, the following definition naturally arises.

Definition 6.2.3. If M is a simple golden-mean matroid with a spanning clique N and an element e such that $M \setminus e$ contains N as a restriction, then e is either in a three-point line or a four-point line ℓ , such that $\ell \setminus \{e\} \subseteq E(N)$. If ℓ is a three-point line we call e a **red element**, and if ℓ is a four-point line we call e a **green element**.

We would like to be able to describe the green and red elements of M in an useful way – this leads us to the notion of an augmented clique.

Definition 6.2.4. An **augmented clique** (G, \mathcal{A}) is a complete graph G with a multiset \mathcal{A} of sets of edges of G , where each member of \mathcal{A} either induces a K_3 in G or is a two-edge matching of G . In the former case we call the member of \mathcal{A} a **green triangle**, and in the latter case we call the member of \mathcal{A} a **red matching**.

Definition 6.2.5. Let $C = (G, \mathcal{A})$ be an augmented clique. A **restriction** of C is an augmented clique $C' = (G', \mathcal{A}')$ such that

- (i) G' is a complete subgraph of G ,

- (ii) $\mathcal{A}' \subseteq \mathcal{A}$, and
- (iii) all members of \mathcal{A}' consist of edges of G' .

In a 3-connected golden-mean matroid M with a spanning clique N , the elements of $E(M) \setminus E(N)$ can be partitioned into a green block and a red block. Each element in the green block is in a line with the edges of a green triangle, and each element in the red block is in a line with the edges of a red matching. Thus every golden-mean matroid with a spanning clique gives rise to an augmented clique.

Definition 6.2.6. Every simple golden-mean matroid with a spanning clique corresponds to at least one augmented clique. Any augmented clique that arises in such a manner is a *golden-mean augmented clique*.

We denote augmented cliques as coloured graphs. These graphs are simple, though we draw them as non-simple graphs to ease comprehension. In other words, if an edge e is contained in a collection of coloured elements, we draw e as a parallel class, with one edge corresponding to each coloured element. For an example, see Figure 6.7.1. We also use the language of non-simple graphs at times.

Note that, as a matroid need not have a unique spanning clique, a given matroid may not give rise to a unique augmented clique.

Not every augmented clique corresponds to a golden-mean matroid. We call such configurations forbidden.

Definition 6.2.7. Let C be an augmented clique. We say that C is *forbidden* if there is no simple golden-mean matroid with a spanning clique, such that the corresponding augmented clique is equivalent to C .

6.3 Forbidden Configurations

If we have a collection of green triangles and red matchings that is not golden-mean, then any augmented clique containing such a configuration as a subgraph will also not be golden-mean. As such, it is useful to find forbidden configurations of green triangles and red matchings.

The next lemma, whose proof is trivial, states that restrictions of augmented cliques work as one would expect.

Lemma 6.3.1. *Assume M is a golden-mean matroid with spanning clique $M(K)$, and let (K, \mathcal{A}) be the corresponding augmented clique. Let K' be a complete subgraph of K . Let X be the closure of $E(K')$ in M . Then $M|X$ is a golden-mean matroid with a spanning clique, and the corresponding augmented clique is equal to the restriction of (K, \mathcal{A}) to K' .*

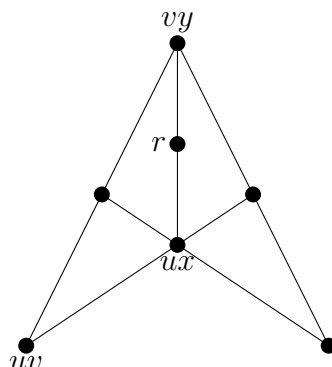
We also need to contract elements of golden-mean augmented cliques.

Lemma 6.3.2. *Let (G, \mathcal{A}) be a golden-mean augmented clique with corresponding golden-mean matroid M , and let $e = uv$ be an edge of G . Let w be the vertex produced by identifying u and v in G/e . The golden-mean matroid $\text{si}(M/e)$ corresponds to (G', \mathcal{A}') , where $G' = \text{si}(G/e)$, and \mathcal{A}' consists of the following green triangles and red matchings, up to swapping the labels of u and v .*

- (i) *If X is a member of \mathcal{A} that does not touch u , then X is a member of \mathcal{A}' .*
- (ii) *If $T = uxy$ is a member of \mathcal{A} , and $T' = vxy$ is not a member of \mathcal{A} , then the green triangle $T = wxy$ is a member of \mathcal{A}' .*
- (iii) *If $T = uxy$ is a member of \mathcal{A} , and $T' = vxy$ is also a member of \mathcal{A} , and the green elements corresponding to T and T' (for any such T') in M/e are not parallel, then the green triangle $T = wxy$ is a member of \mathcal{A}' .*
- (iv) *If $R = (ux, vy)$ is a red matching from \mathcal{A} , then the green triangle $T = wxy$ is a member of \mathcal{A}' , unless this would induce a parallel pair in M/e .*

Proof. Statements (i), (ii), and (iii) are straightforward. To confirm (iv), consider the K_4 spanned by R . We have shown this in Figure 6.1. The element of R is labelled r , and it is on the line spanned by ux and vy . The element $e = uv$ is an element inside the K_4 -restriction spanned by R . Now, when we contract uv and simplify, we see a matroid isomorphic to $U_{2,4}$, with groundset $\{r, vy, wx, a\}$, where a is an element from the K_4 spanned by R . The result now follows. \square

All the verifications will be done by Sage [19], by attempting to put all the new elements on top of a complete graph. To optimise this effort, we only

Figure 6.1: F_7^- for Lemma 6.3.2

check one possible location for the configuration. This is acceptable because of the myriad symmetries in complete graphs.

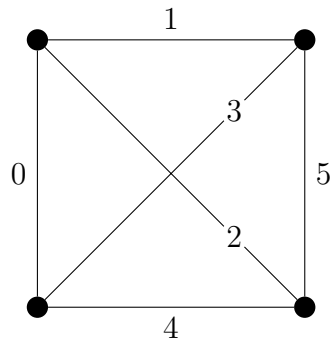
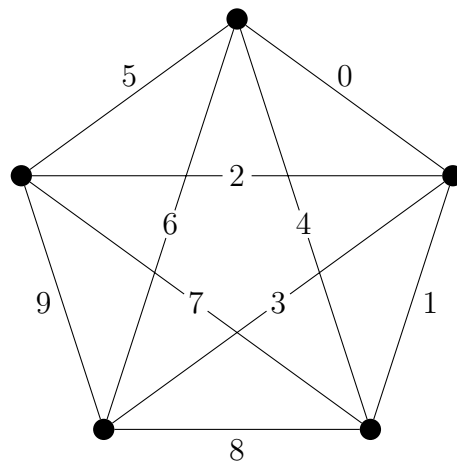
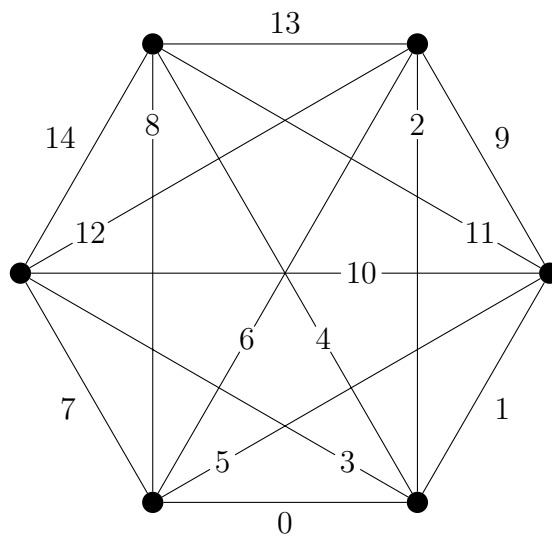
Note that throughout this chapter, when we say that there are x ways to place an element into a clique, we mean x ways, up to whatever symmetries are natural. We make frequent and unstated use of this fact throughout this chapter.

6.3.1 Complete Graphs

In order to select a location for the configurations to go, we need to have labelled versions of all the complete graphs we are going to build from. The edge labels for K_3 are not given as there is only one way to label K_3 . The edge labels for K_4 are shown in Figure 6.2, the edge labels for K_5 are shown in Figure 6.3, the edge labels for K_6 are shown in Figure 6.4, the edge labels for K_7 are shown in Figure 6.5, and the edge labels for K_8 are shown in Figure 6.6. These labels come from the standard construction in Sage of the relevant matroid.

6.3.2 Forbidden Configurations

First up, we have TR, MR, and PN, shown in Figure 6.7. Next is QI, OM, TL, TB, VQ, and RA, shown in Figure 6.8. Then we have BL, ML, TT, EU, and DJ, shown in Figure 6.9. Next is XS, TM, BC, MW, and DM, shown in Figure 6.10. Following is XG, AE, AF, PL, SW, and DI, shown in

Figure 6.2: Edge Labels for K_4 Figure 6.3: Edge Labels for K_5 Figure 6.4: Edge Labels for K_6

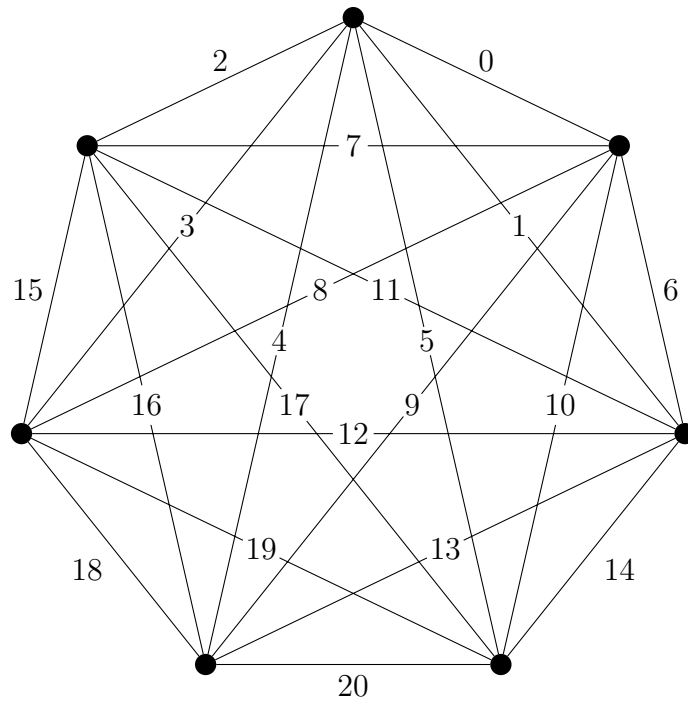
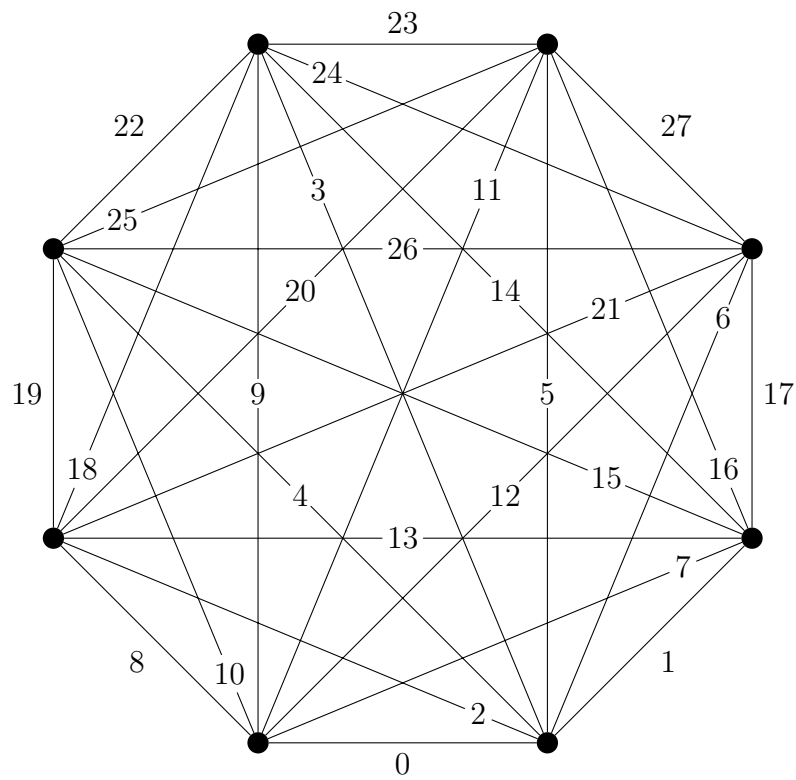
Figure 6.5: Edge Labels for K_7 Figure 6.6: Edge Labels for K_8

Figure 6.11. Then we have TP, KF, PM, CC, and CF, shown in Figure 6.12. Finally, we have HN, FS, WB, HM, SR, and BP, shown in Figure 6.13.

We need to verify that these configurations are indeed forbidden.

Lemma 6.3.3. *All of the configurations given in Figures 6.7 – 6.13 are forbidden.*

Proof. We run each configuration through the function `is_forbidden()`, shown in Appendix 6.A.3. The calls to the function are given here. The numbers in the list come from the appropriate labelling of K_n . For example, to get the numbers for TR, we look at Figure 6.7.1, and note that TR is on five vertices, with one green triangle and one red matching. We now look at the labels for K_5 , Figure 6.3, and note that the triangle in TR corresponds to $\{0, 2, 5\}$, and the matching to $\{2, 8\}$.

```

TR is_forbidden(5, [[0, 2, 5], [2, 8]])
MR is_forbidden(6, [[1, 2, 9], [7, 13]])
PN is_forbidden(7, [[0, 2, 7], [14, 18]])
QI is_forbidden(3, [[0, 1, 2], [0, 1, 2], [0, 1, 2]])
OM is_forbidden(5, [[0, 2, 5], [1, 3, 8], [7, 8, 9]])
TL is_forbidden(6, [[1, 2, 9], [7, 8, 14]])
TB is_forbidden(6, [[0, 3, 7], [1, 2, 9], [12, 13, 14]])
VQ is_forbidden(6, [[0, 4, 8], [1, 4, 11], [7, 8, 14], [9, 11,
    13], [12, 13, 14]])
RA is_forbidden(7, [[0, 1, 6], [2, 3, 15], [4, 5, 20]])
BL is_forbidden(6, [[1, 13], [4, 7]])
ML is_forbidden(6, [[1, 7], [1, 13]])
TT is_forbidden(6, [[0, 9], [7, 13]])
EU is_forbidden(7, [[0, 14], [2, 18]])
DJ is_forbidden(8, [[1, 22], [8, 27]])
XS is_forbidden(4, [[0, 5], [1, 4], [2, 3]])
TM is_forbidden(4, [[1, 4], [1, 4], [1, 4]])
BC is_forbidden(5, [[1, 5], [1, 9], [5, 8]])
MW is_forbidden(5, [[0, 8], [1, 9], [1, 9]])
DM is_forbidden(5, [[0, 9], [1, 9], [5, 8]])
XG is_forbidden(5, [[1, 9], [3, 7], [5, 8]])
AE is_forbidden(5, [[0, 7], [0, 8], [0, 9]])

```

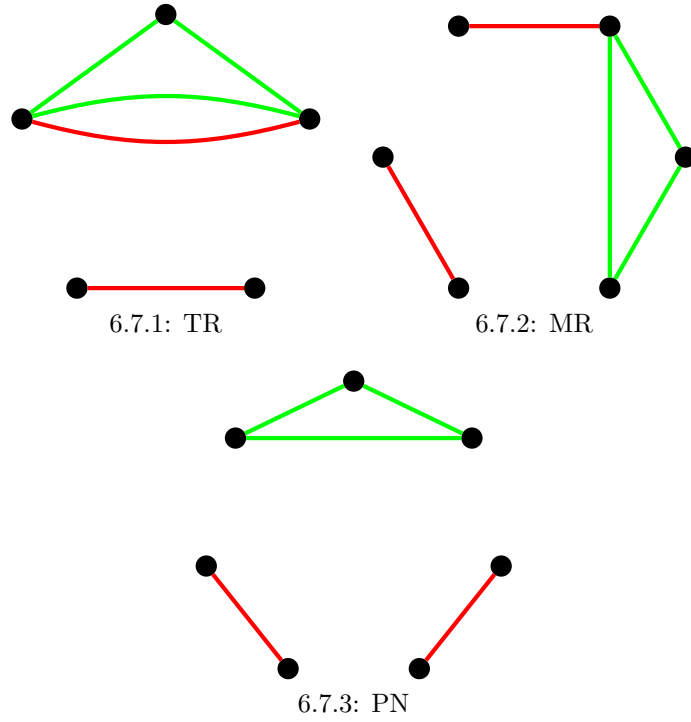


Figure 6.7: Forbidden configurations with one green triangle and one red matching

```

AF is_forbidden(5, [[0, 7], [0, 7], [0, 8]])
PL is_forbidden(5, [[0, 7], [0, 9], [2, 8]])
SW is_forbidden(5, [[1, 9], [2, 4], [5, 8]])
DI is_forbidden(6, [[0, 14], [7, 13], [8, 10]])
TP is_forbidden(4, [[0, 2, 4], [0, 2, 4], [1, 4], [1, 4]])
KF is_forbidden(5, [[0, 1, 4], [1, 3, 8], [6, 7]])
PM is_forbidden(5, [[0, 2, 5], [1, 3, 8], [2, 4]])
CC is_forbidden(5, [[1, 3, 8], [1, 3, 8], [0, 9]])
CF is_forbidden(5, [[0, 3, 6], [1, 9], [1, 9]])
HN is_forbidden(5, [[1, 2, 7], [1, 9], [5, 8]])
FS is_forbidden(5, [[0, 3, 6], [1, 9], [5, 8]])
WB is_forbidden(5, [[0, 3, 6], [1, 5], [1, 9]])
HM is_forbidden(5, [[7, 8, 9], [1, 6], [5, 8]])
SR is_forbidden(6, [[0, 3, 7], [2, 4, 13], [1, 14]])
BP is_forbidden(6, [[5, 8, 11], [0, 14], [7, 13]])

```

□

Lemma 6.3.4. *In a golden-mean augmented clique, every edge of every red*

matching must touch every green triangle.

Proof. We will prove this using the three forbidden configurations in Figure 6.7.

Assume that there is a red matching that has an edge that does not touch a green triangle. If the matching is vertex-disjoint from the triangle, we can find a K_7 -restriction of the clique that is a PN (Figure 6.7.3). So only one edge of the matching must be incident with the triangle. This edge can either be incident with the triangle in one vertex (MR, Figure 6.7.2) or two vertices (TR, Figure 6.7.1).

As all other possibilities are forbidden, the statement is true. \square

Lemma 6.3.5. *In a golden-mean augmented clique, every edge of every red matching must touch every other red matching.*

Proof. We will prove this using the forbidden configurations in Figure 6.9.

Assume not. So we have a red matching, and we are placing a second red matching so that there is an edge in the second red matching vertex-disjoint from both edges in the first red matching. This can only be done in one way. Now we try to place the final edge of the second red matching. If we place it parallel with an existing red matching edge, we get a ML (Figure 6.9.2). If we place it so it only that it is incident with an existing red matching edge at a single vertex, we get an EU (Figure 6.9.4). If we place it so that it is vertex-disjoint from the existing red matching, we get a DJ (Figure 6.9.5). So it must be incident with both existing red matching edges. There is only one way to do this, which is BL (Figure 6.9.1).

As all other possibilities have been exhausted, the statement is true. \square

6.4 Non-clique Three-point Lines

In order to prove Theorem 6.1.4, we need to understand how non-clique three-point lines and $U_{2,4}$ -restrictions behave in the presence of a spanning clique. We start by analysing the non-clique three-point lines.

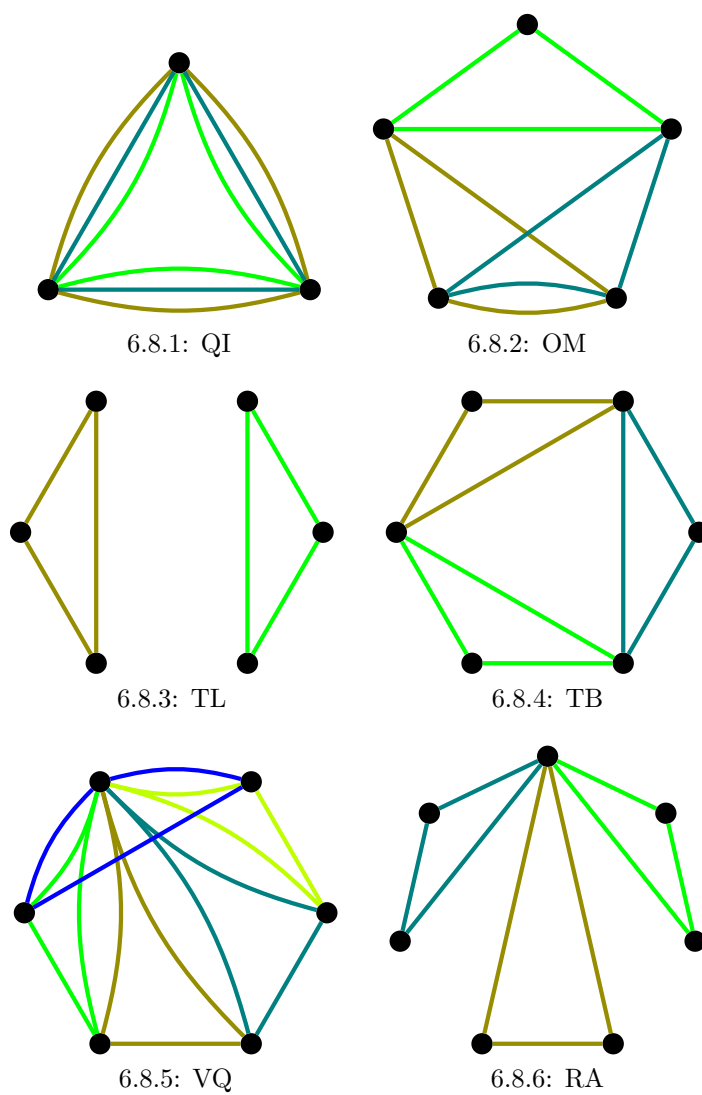


Figure 6.8: Forbidden configurations with only green triangles

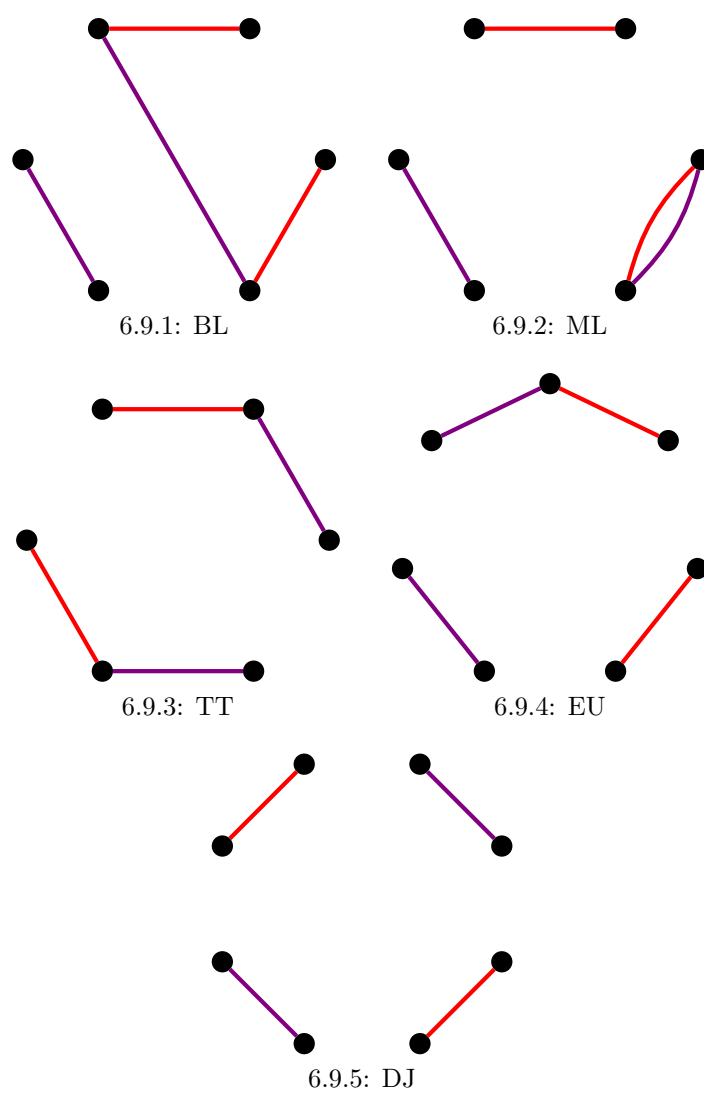


Figure 6.9: Forbidden configurations with two red matchings

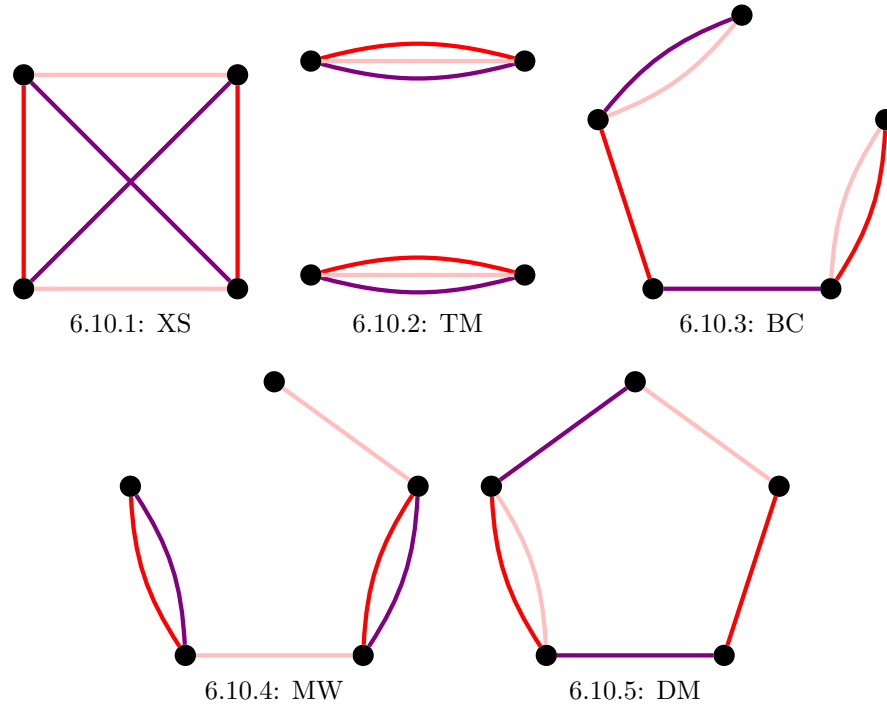


Figure 6.10: Forbidden configurations with three red matchings I

6.4.1 One Non-clique Three-point Line

We need to analyse the ways that a clique element can be on a non-clique three-point line.

The non-clique three-point line could have either one or two non-clique elements on it. If it has two non-clique elements, these could be both green, both red, or one red and one green. We consider each in turn, by adding the coloured elements in all possible ways, and then checking to see whether the non-clique elements form a triangle with any clique elements.

6.4.1.1 One Non-clique Element

There are two ways to add in one non-clique element: a red matching or a green triangle. The red matching does create a non-clique three-point line, while the green triangle does not, as lines are flats.

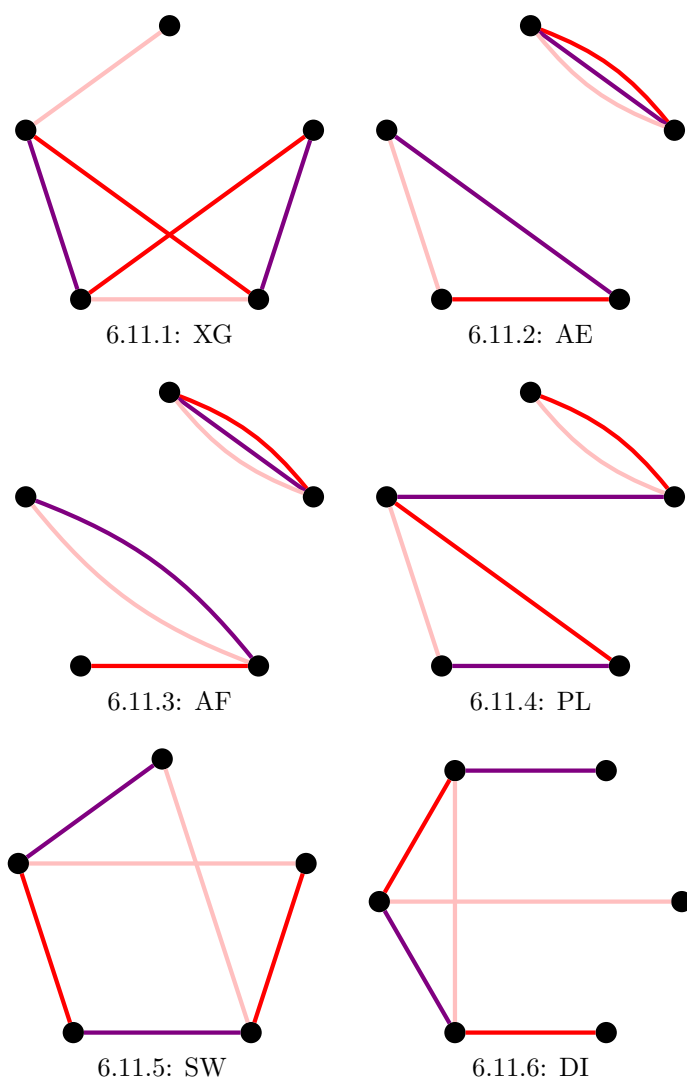


Figure 6.11: Forbidden configurations with three red matchings II

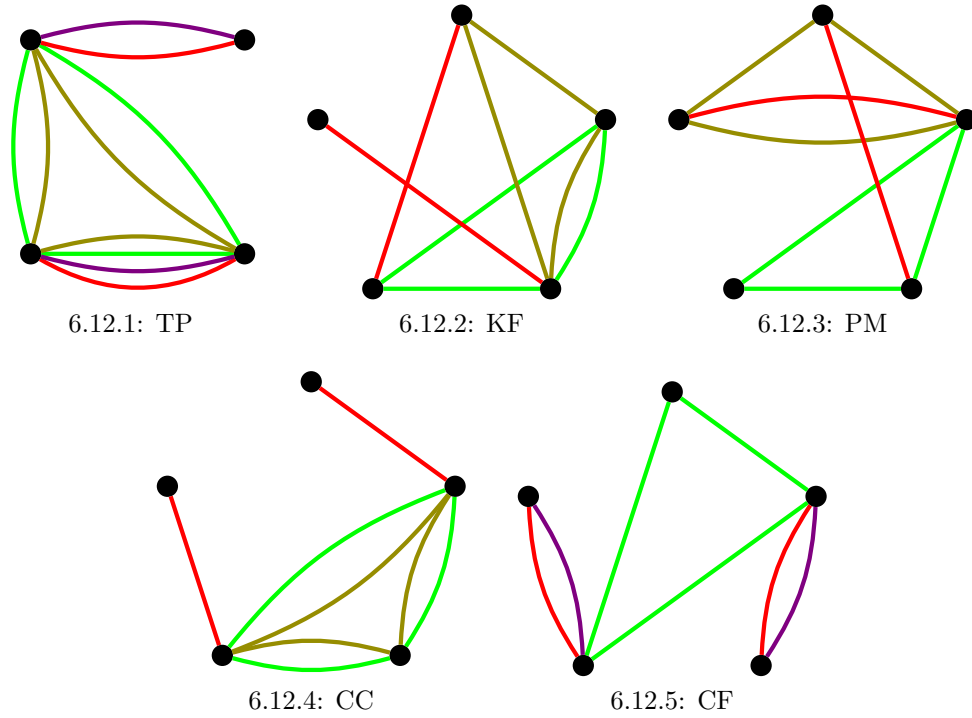


Figure 6.12: Remaining forbidden configurations I

6.4.1.2 Two Green Triangles

Because TL (Figure 6.8.3) is forbidden, the two triangles must not be vertex-disjoint. This leads to three possible configurations: the two triangles meet at a triangle (Figure 6.14.1), an edge (Figure 6.14.2), or a vertex (Figure 6.14.3).

We now go through each configuration in turn and check for the existence of non-clique three-point lines containing both of the extension elements. We also look for $U_{2,4}$ -restrictions containing both of the extension elements, as these are required later, in Section 6.5.2. These checks are done by the `lines()` function from Appendix 6.A.4. We are able to use the smallest spanning clique possible because any line containing the green and red elements must be in the span of said elements. The function calls, and an interpretation of the output are given below.

Triangle `lines(3, [[0, 1, 2], [0, 1, 2]])` outputs that the only line containing both extension elements is the expected five-point line.

Edge `lines(4, [[0, 2, 4], [1, 2, 5]])` outputs that `[a, b, 3]` is a non-clique three-point line containing both extension elements. This

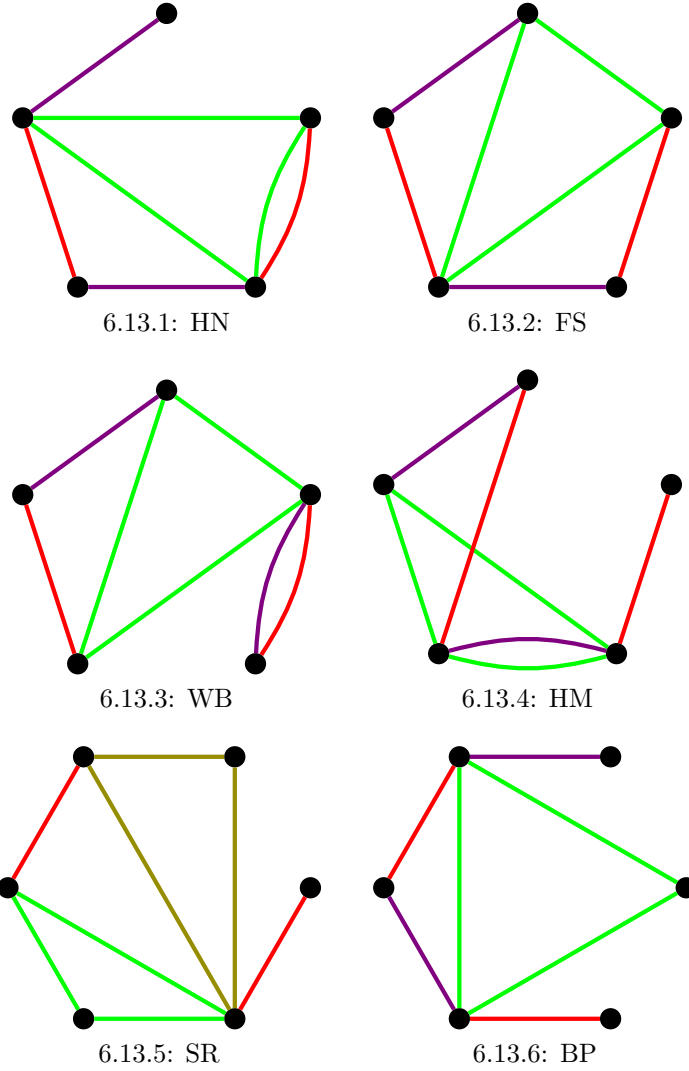


Figure 6.13: Remaining forbidden configurations II

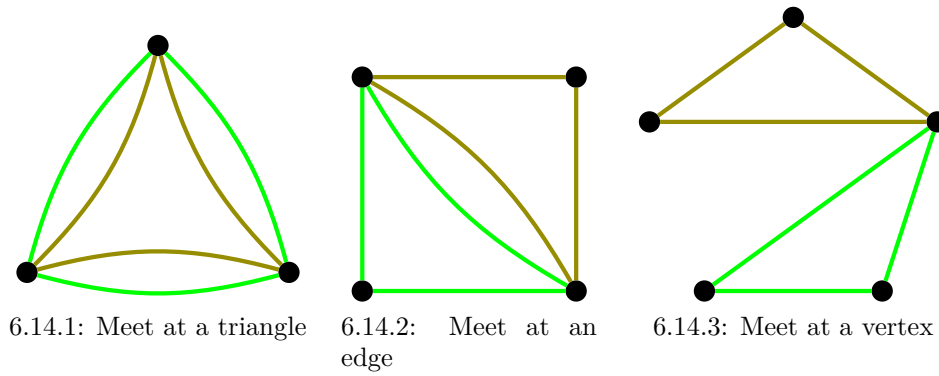


Figure 6.14: Two Green Triangles

is GG, shown in Figure 6.17.1.

Vertex `lines(5, [[0, 2, 5], [1, 3, 8]])` outputs that there is no line containing both extension elements.

6.4.1.3 Two Red Matchings

We further break the case checking up by number of vertices. First, we analyse all the possible configurations, and then we test each of them in turn for the required properties.

Four vertices We do this systematically. Place the first red matching, there is only one way to do this. Now place the first edge of the second red matching. It can either join the existing red matching edges or it could be parallel to the existing red matching. In both cases there is only one way to put in the final edge. In the first case we get a square of red matchings (Figure 6.15.1), and in the second case we get a double red matching (Figure 6.15.2).

Five vertices As before, there is only one way to place the first red matching, which uses four of the vertices. The first edge of the final red matching must touch the unused vertex, and then there is only one place for it to go. There are now two ways we can add in the final edge: either by joining the two ends of the first red matching (Figure 6.15.3), or by placing an edge in parallel (Figure 6.15.4).

Six vertices The two vertices untouched by the first red matching cannot be joined by an edge in the second red matching, as this would contradict Lemma 6.3.5. This leaves only one way that the second red matching can be added. This is shown in Figure 6.15.5, which is a forbidden configuration (TT, Figure 6.9.3).

Seven or more vertices This is impossible due to Lemma 6.3.5, as the second red matching will need to touch at least three vertices that the first red matching does not touch, meaning at least one edge must be disjoint.

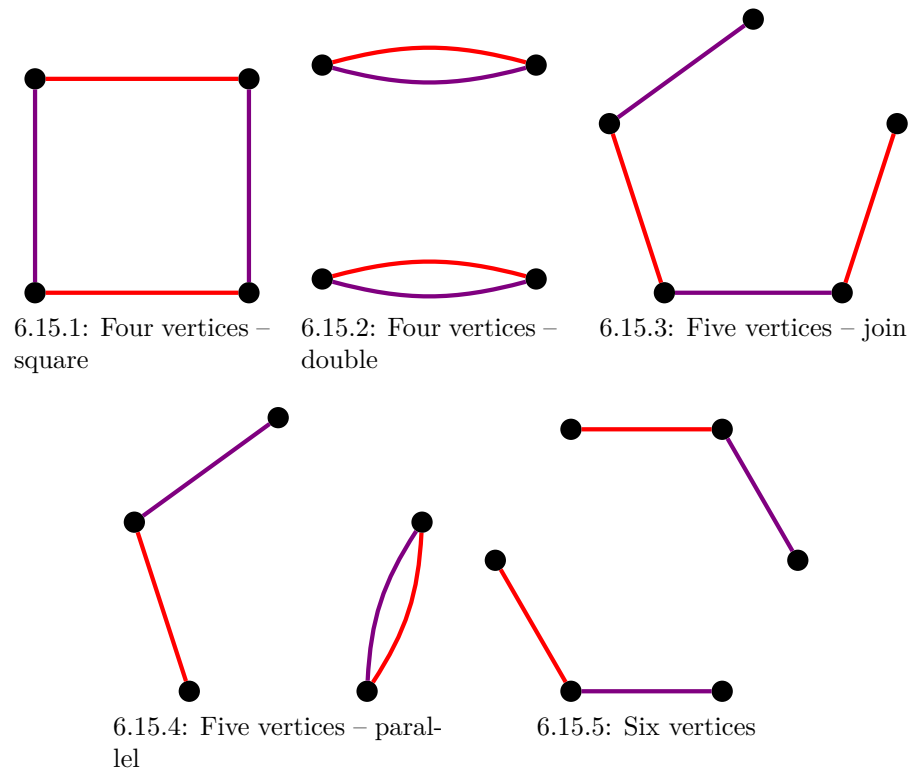


Figure 6.15: Two Red Matchings

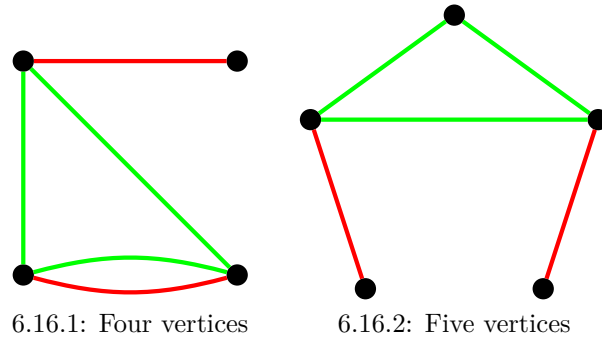


Figure 6.16: One Green Triangle and One Red Matching

We now go through each configuration in turn and check for the existence of non-clique three-point lines containing both of the extension elements. We also look for $U_{2,4}$ -restrictions containing both of the extension elements, as these are required later, in Section 6.5.2. These checks are done by the `lines()` function from Appendix 6.A.4. The function calls, and an interpretation of the output are given below.

Square `lines(4, [[0, 5], [1, 4]])` outputs that `[a, b, 3]` is a non-clique three-point line containing both extension elements. This is RR-square, shown in Figure 6.17.3.

Double `lines(4, [[1, 4], [1, 4]])` outputs that the only line containing both extension elements is the expected four-point line.

Join `lines(5, [[1, 9], [5, 8]])` outputs that there is no line containing both extension elements.

Parallel `lines(5, [[1, 5], [1, 9]])` outputs that `[a, b, 6]` is a non-clique three-point line containing both extension elements. This is RR-digon, shown in Figure 6.17.4

6.4.1.4 One Red Matching and One Green Triangle

By Lemma 6.3.4, both edges of the matching must touch the triangle. This leads to only two possible configurations, one on four vertices (Figure 6.16.1) and one on five vertices (Figure 6.16.2).

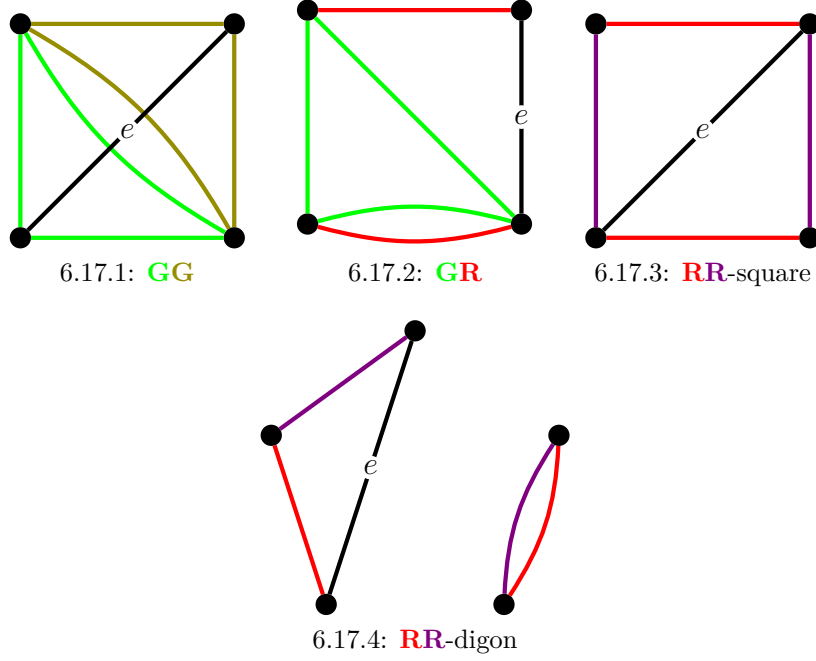


Figure 6.17: One Non-clique Three-point line

We now go through each configuration in turn and check for the existence of non-clique three-point lines containing both of the extension elements. We also look for $U_{2,4}$ -restrictions containing both of the extension elements. These checks are done by the `lines()` function.

Four `lines(4, [[0, 2, 4], [1, 4]])` outputs that `[a, b, 5]` is a non-clique three-point line containing both extension elements. This is GR, shown in Figure 6.17.2.

Five `lines(5, [[0, 2, 5], [1, 9]])` outputs that there is no line containing both extension elements.

6.4.1.5 Summary

In summary, there are five ways that a clique element can be on a non-clique three-point line while not being in a $U_{2,4}$ -restriction. Firstly, we could make the clique element part of a red matching. The other four configurations, GG, GR, RR-square, and RR-digon, are shown in Figure 6.17, where e is the distinguished clique element.

As we will be referring to them later, note that there are two components of

a RR-digon: a *parallel component* (the two edges in a parallel class), and a *path component* (the two edges that touch e).

6.4.2 Two Non-clique Three-point Lines

We now want to find all the ways that an element e of the clique can be on two non-clique three-point lines, without being in a $U_{2,4}$ -restriction. We have the four configurations in Figure 6.17, as well as the ability to make e part of a red matching, and need to consider the ways to combine these five configurations in order to put e on two non-clique three-point lines. We consider each case in turn.

At no point can we put e into a green triangle, as this would put e onto a four-point line, which we are avoiding.

We use the function `all_lines()`, described in Appendix 6.A.5 to find all the lines that e is on.

6.4.2.1 GG

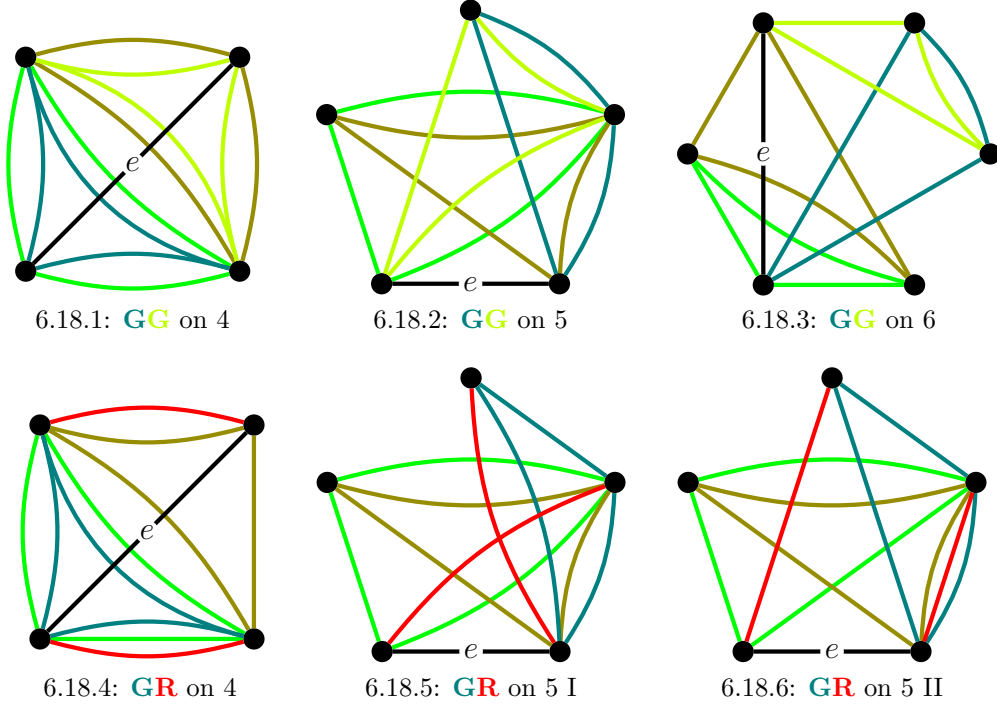
Firstly, we start with a GG (Figure 6.17.1), and add in the second line in each of the possible ways.

GG We first consider the case where both lines are GGs.

If we remain on four vertices, there is only one way we can put the second line in, shown in Figure 6.18.1. In this case, `all_lines(4, [[0, 2, 4], [0, 2, 4], [1, 2, 5], [1, 2, 5]], 3)` outputs that e is on two non-clique three-point lines. We call this configuration Γ (Figure 6.24.1).

If we add in one extra vertex, there is also only one way we can put the second line in, shown in Figure 6.18.2. In this case, `all_lines(5, [[0, 1, 4], [0, 3, 6], [1, 2, 7], [2, 3, 9]], 8)` outputs that e is on two non-clique three-point lines. We call this configuration Δ (Figure 6.24.2).

If we add in two extra vertices, there is also only one way we can put the second line in, shown in Figure 6.18.3. In this case, we can see a TL (Figure 6.8.3), so this configuration is forbidden. As GG has only two vertices disjoint from e , this is the largest number of vertices that we can add in.

Figure 6.18: Starting from a **GG** I

GR We now start with a GG, and make the second line a GR (Figure 6.17.2).

If we remain on four vertices, there is only one way we can put the second line in, shown in Figure 6.18.4. In this case, `all_lines(4, [[0, 2, 4], [0, 2, 4], [1, 2, 5], [1, 4]], 3)` outputs that e is on a four-point line, so we discard this case.

If we add in one extra vertex, then there is only one location for the green triangle in the GR. After this is placed, there are two places we could put the red matching, shown in Figures 6.18.5 and 6.18.6. Both configurations contain a red matching edge disjoint from a triangle, and hence are forbidden by Lemma 6.3.4.

If we add in two extra vertices, there is only one way we can put the second line, shown in Figure 6.19.1. This contains a matching edge disjoint from a triangle, and hence is forbidden by Lemma 6.3.4.

RR-square We now start with a GG, and make the second line a RR-square (Figure 6.17.3). We also take Lemma 6.3.4 into account, as it reduces

the number of locations that a matching edge could go.

Regardless of how many vertices we have, the red matching edges must all be incident with both green triangles. Furthermore, each red matching edge must also be incident with e . This leaves only one possible location for the RR-square to go, shown in Figure 6.19.2. In this case, `all_lines(4, [[0, 2, 4], [1, 3, 5], [0, 5], [1, 4]], 3)` outputs that, among other cases, it is possible for e to be on two non-clique three-point lines. We call this configuration Θ (Figure 6.24.3).

RR-digon We now start with a GG, and make the second line a RR-digon (Figure 6.17.4).

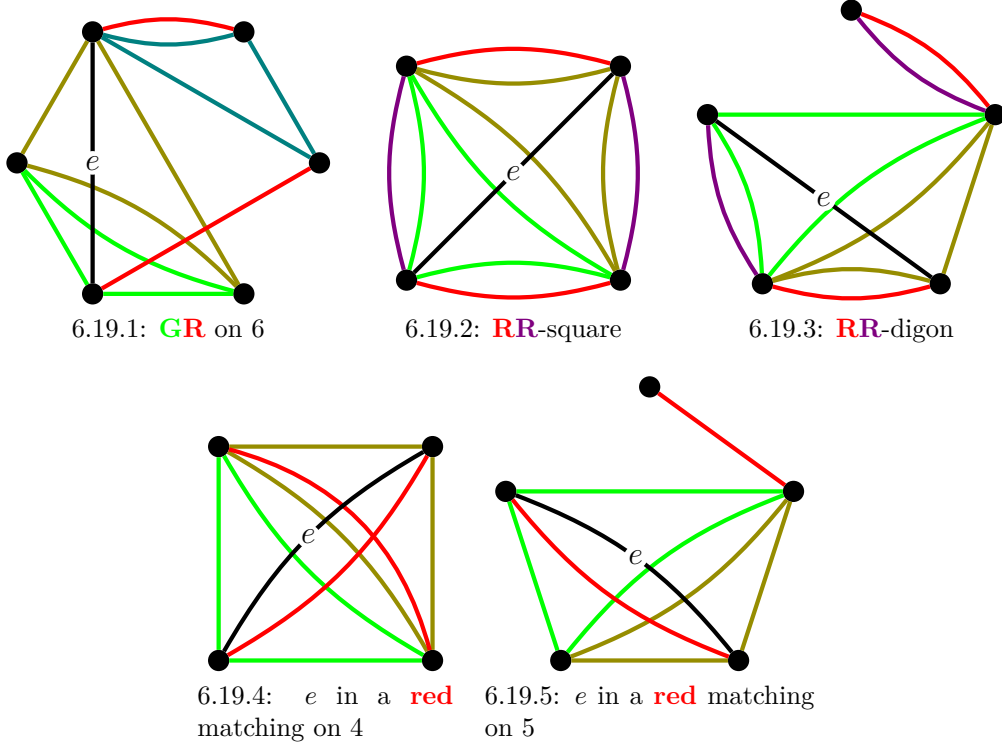
No matter how many vertices we have, the red matching edges must all be incident with both green triangles. As the edges of the path component of the RR-digon must also be incident with e , there is only one place that they can go. This leaves one vertex that the parallel class must be incident with, giving the configuration shown in Figure 6.19.3. In this case, `all_lines(5, [[1, 3, 8], [2, 3, 9], [0, 8], [0, 9]], 7)` outputs that e is on two non-clique three-point lines. We call this configuration Λ (Figure 6.24.4).

e in a red matching We now start with GG, and form the second non-clique three-point line by making e part of a red matching.

The first matching edge must go parallel with e . As the last matching edge must be incident with both green triangles, there are two possible configurations. The first configuration, on four vertices, is shown in Figure 6.19.4. In this case, `all_lines(4, [[0, 2, 4], [1, 3, 5], [2, 3]], 3)` outputs that, among other cases, it is possible for e to be on two non-clique three-point lines. We call this configuration Ξ (Figure 6.24.5). The second configuration, on five vertices, is shown in Figure 6.19.5. This is KF (Figure 6.12.2), and so is forbidden.

6.4.2.2 GR

We now start from a GR (Figure 6.17.2) and add in the second line in each of the possible ways, with the exception of GG, as that case has already been

Figure 6.19: Starting from a GG II

covered.

GR First, we consider the case where both lines are GRs. If this is on four vertices, we can either place the second GR on top of the first GR, as shown in Figure 6.20.1, or flip it over the perpendicular bisector of e , as shown in Figure 6.20.2. In the first case, we note that it is TP (Figure 6.12.1), and so is forbidden. In the second case, `all_lines(4, [[0, 2, 4], [3, 4, 5], [0, 5], [0, 5]], 1)` outputs that e is on two non-clique three-point lines. We call this configuration II (Figure 6.24.6).

We now consider the case where both lines are GRs, and we are on five vertices. The second green triangle can either be incident with the vertex where the first green triangle touches e , or it could not. If it is, as each green triangle must touch every matching edge, there is only one location for it. Furthermore, out of the two possible locations of the red matching, only one does not contradict Lemma 6.3.4, and so the only possible configuration is shown in Figure 6.20.3. Now, if the second green triangle is not incident with

the vertex where e meets the first green triangle, by similar reasoning, there is also only one place for the second GR to go, shown in Figure 6.20.4. In the first case, `all_lines(5, [[0, 3, 6], [1, 3, 8], [0, 9], [1, 9]], 2)` outputs that e is on no non-clique three-point lines, so this case is discarded. In the second case, this configuration contains an HM (Figure 6.13.4), and so is forbidden.

When we try to place two GRs on six vertices, we quickly arrive at a TT (Figure 6.9.3), and so this is forbidden.

RR-square We now consider the case where the first line is a GR and the second line is a RR-square (Figure 6.17.3). On four vertices, there is only one way to do this, shown in Figure 6.20.5. In this case, `all_lines(4, [[0, 2, 4], [1, 4], [1, 4], [2, 3]], 5)` outputs that e is on a four-point line, so this case is discarded. If we try to do the same pairing on five (or more) vertices, there must be a matching edge incident with a new vertex and the vertex that is incident with e but not incident with the existing green triangle. This matching edge is vertex-disjoint from the existing green triangle, a contradiction to Lemma 6.3.4.

RR-digon The next case is where the first line is a GR and the second line is a RR-digon (Figure 6.17.4). By the same reasoning as in the previous case, the path component of the digon must be in the span of the GR, and hence there are two possible locations for the path. As the parallel component must be incident with the green triangle, once the path is placed, there is only one location remaining for the parallel component. These two configurations are shown in Figures 6.20.6 and 6.20.7. In each configuration, we can find a BC (Figure 6.10.3), and so both configurations are forbidden.

e in a red matching Finally, we start with a GR, and make e part of a red matching. On four vertices, there is only one way to do this, shown in Figure 6.20.8. In this case, `all_lines(4, [[0, 2, 4], [0, 5], [1, 4]], 5)` outputs that, among other cases, e is on two non-clique three-point lines. We call this configuration Σ (Figure 6.24.7). We now do this on five vertices. The last red matching edge can either be incident with the edge that both the existing green triangle and existing red matching use (Figure 6.20.9), or it

could not be (Figure 6.20.10). In the first case, this is a HM (Figure 6.13.4), and in the second case this is a HN (Figure 6.13.1). So both cases are forbidden.

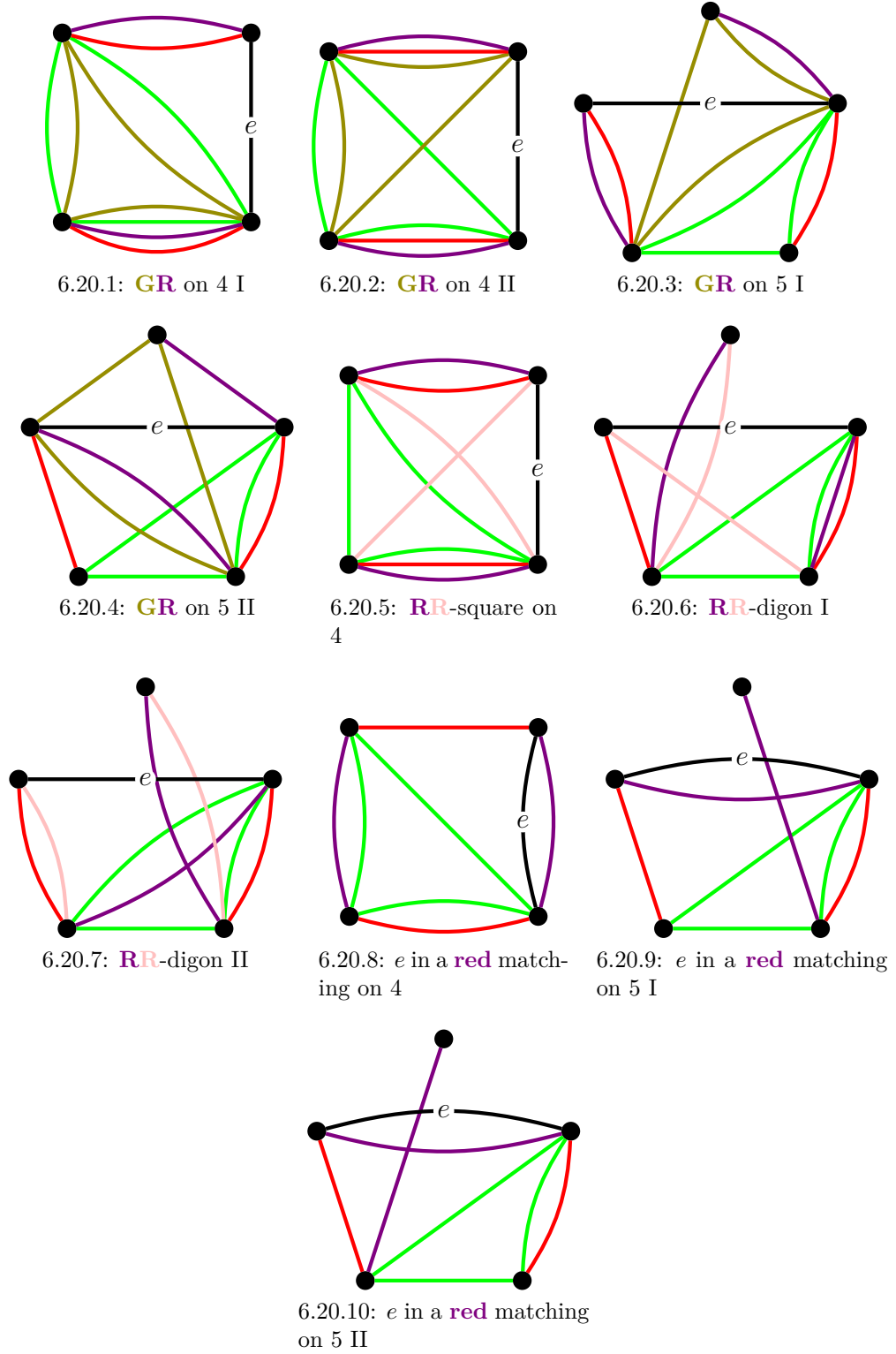
6.4.2.3 RR-square

We now start from a RR-square (Figure 6.17.3) and add in the second line in each of the possible ways, with the exception of GG and GR, as those cases have already been covered.

RR-square We start with both lines being a RR-square. There is only one way to do this on each of four (Figure 6.21.1), five (Figure 6.21.2), and six (Figure 6.21.3) vertices. In the four vertex case, `all_lines(4, [[0, 5], [0, 5], [1, 4], [1, 4]], 3)` outputs that e is on two non-clique three-point lines. We call this configuration Υ (Figure 6.24.8). In the five vertex case, `all_lines(5, [[0, 8], [1, 6], [1, 9], [2, 8]], 3)` outputs that e is on two non-clique three-point lines. We call this configuration Φ (Figure 6.24.9). In the six vertex case, we are able to see a TT (Figure 6.9.3), and so this case is forbidden.

RR-digon The next case is when the first line is a RR-square and the second line is a RR-digon (Figure 6.17.4). If we do this on five vertices, either the parallel or the path component of the RR-digon can be incident with the new vertex. In the first case (Figure 6.21.4), we can find a BC (Figure 6.10.3), and so this is forbidden. In the second case (Figure 6.21.5), we can find a DM (Figure 6.10.5), and so this is forbidden. On six vertices, both the parallel and path components of the RR-digon need to be incident with new vertices, which leaves only one possibility, Figure 6.21.6. In this case, we can see a TT (Figure 6.9.3), and so this case is forbidden.

e in a red matching Finally, we consider the case where the first line is a RR-square and e is part of a red matching. If we are to not contradict Lemma 6.3.5, then we can either do this on four (Figure 6.21.7), or five (Figure 6.21.8) vertices. In the four vertex case, we get XS (Figure 6.10.1), and

Figure 6.20: Starting from a **GR**

so this case is forbidden. In the five vertex case, we get XG (Figure 6.11.1), and so this case is forbidden.

6.4.2.4 RR-digon

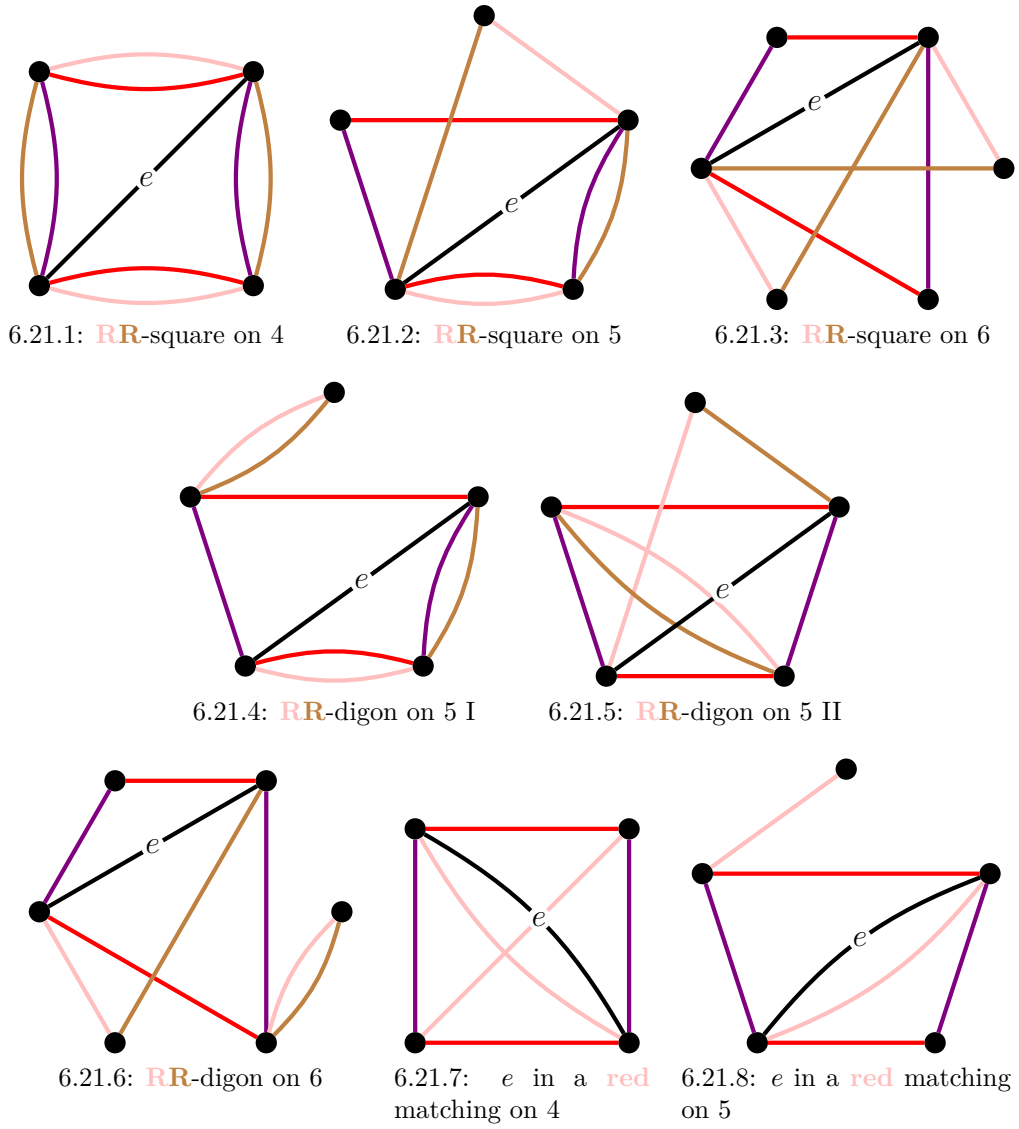
We now start from a RR-digon (Figure 6.17.4) and add in the second line in both of the remaining possible ways, which are as a RR-digon and by having e be in a red matching.

RR-digon First, we consider the case where both lines arise from a RR-digon. We do this case slightly differently from the previous cases.

We start by placing the path component of the second RR-digon. The end vertices of this component must be joined by e . This could go parallel to the existing path component, it could touch the existing parallel component, or it could be vertex-disjoint from the first RR-digon. In this last case, it is possible to find a contradiction to Lemma 6.3.5, so we discard it.

Next, we consider the case where the path component of the second RR-digon is parallel to the path component of the first RR-digon. As the parallel component of the second RR-digon cannot be disjoint from the existing RR-digon by Lemma 6.3.5, there are two locations that we could place it. First, we could place it in parallel to the existing parallel component, giving Figure 6.22.1. In this case, we can find an AF (Figure 6.11.3), and so this case is forbidden. Second, we could place it incident with the existing parallel component, giving Figure 6.22.2. In this case, we can find a TT (Figure 6.9.3), and so this case is forbidden.

Finally, we consider the case where the path component of the second RR-digon touches the existing parallel component. There are now three locations for the parallel component of the second RR-digon to go. First, it could touch only the path component of the first RR-digon, giving Figure 6.22.3. In this case, we can find a TT (Figure 6.9.3), and so this case is forbidden. Secondly, the parallel class of the second RR-digon could touch only the parallel class of the first RR-digon, giving Figure 6.22.4. In this case, the edges in the path component of the first RR-digon and the new parallel class give a contradiction to Lemma 6.3.5, and so this case is forbidden. Finally, we could place the parallel component of the second RR-digon so that it touches

Figure 6.21: Starting from a **RR**-square

both the path and parallel components of the first RR-digon, as shown in Figure 6.22.5. In this case, `all_lines(5, [[1, 5], [1, 9], [2, 4], [2, 8]], 6)` outputs that e is on two non-clique three-point lines. We call this configuration Ψ (Figure 6.24.10).

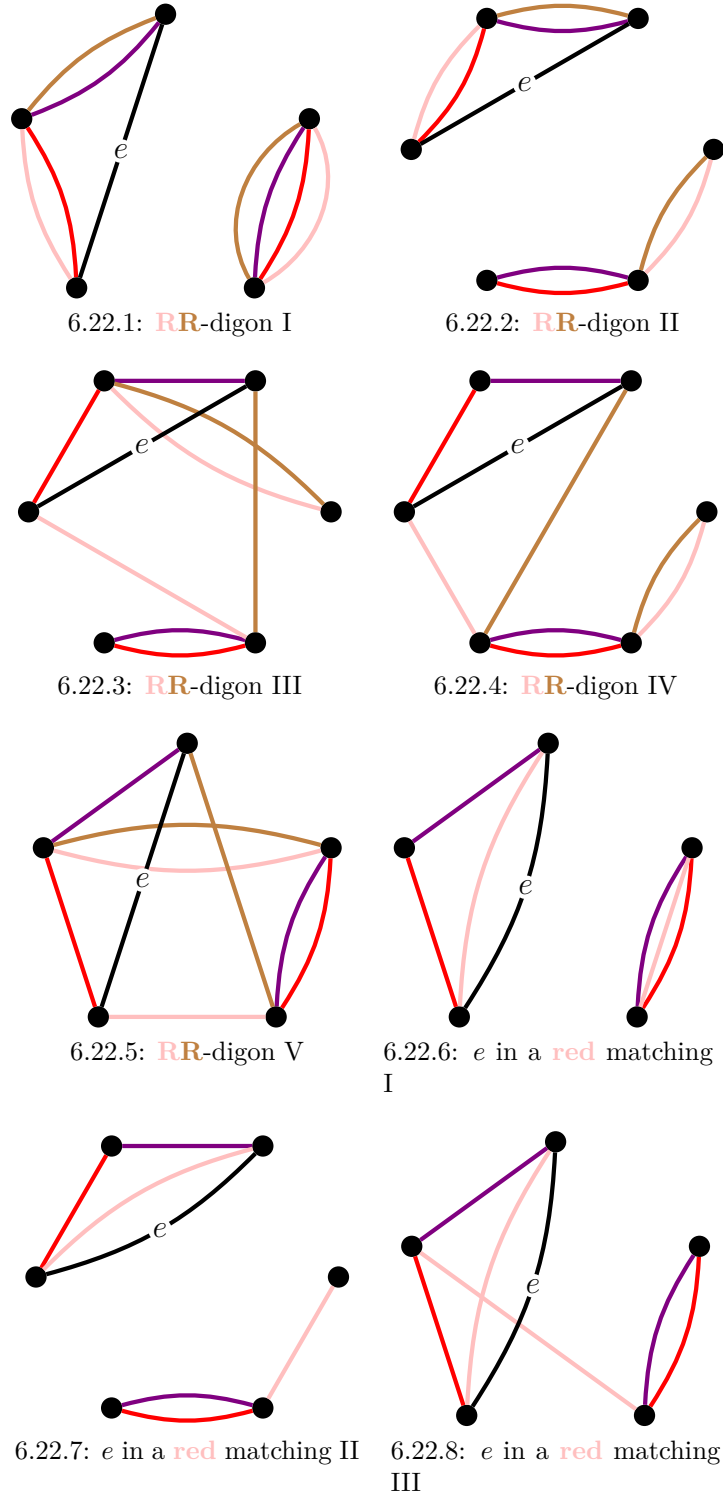
e in a red matching We now consider the case where the first line is a RR-digon and the second line arises from e being part of a red matching. As the second edge of the new red matching cannot avoid the existing parallel component (because of Lemma 6.3.5), there are three places we could place this last edge. Firstly, we could place it so that it is in the existing parallel class, as shown in Figure 6.22.6. This is AE (Figure 6.11.2), and so is forbidden. Secondly, we could place the final edge so that it is incident with the existing parallel class and a vertex outside of the existing RR-digon, as shown in Figure 6.22.7. In this case, we can find a TT (Figure 6.9.3), and so this case is forbidden. Ultimately, we can add the final edge in so that it is incident with both the existing path and parallel components, as shown in Figure 6.22.8. This is PL (Figure 6.11.4), and so is forbidden.

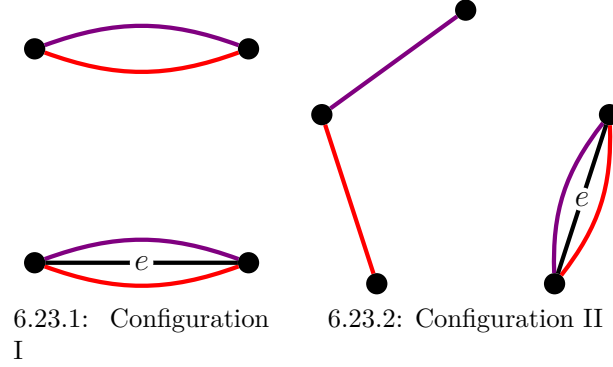
6.4.2.5 e in a matching

The final case to consider is when e is in two red matchings. As matching edges cannot be disjoint by Lemma 6.3.5, there are only two ways to have e in two matchings. The first is to identify the two matchings, shown in Figure 6.23.1. In this case, `all_lines(4, [[1, 4], [1, 4]], 4)` outputs that e is on a four-point line, so this case is discarded. The final configuration is when the two matching edges not parallel with e form a path, as shown in Figure 6.23.2. In this case, `all_lines(5, [[1, 5], [1, 9]], 1)` outputs that e is on two non-clique three-point lines. We call this configuration Ω (Figure 6.24.11).

6.4.2.6 Conclusion

In summary, there are eleven ways that e can be on two non-clique three-point lines: Γ , Δ , Θ , Λ , Ξ , Π , Σ , Υ , Φ , Ψ , and Ω . These are shown in Figure 6.24.

Figure 6.22: Starting from a **RR**-digon

Figure 6.23: Starting from e in a **red** matching

	GG	GR	RR-square	RR-digon	e red
GG	Γ, Δ		Θ	Λ	Ξ
GR		Π			Σ
RR-square	Θ		Υ, Φ		
RR-digon	Λ			Ψ	
e red	Ξ	Σ			Ω

Table 6.1: How the eleven ways to get e on two non-clique three-point lines (Figure 6.24) arise from the five ways to have e on one non-clique three-point line (Figure 6.17 and e being in a red matching).

6.4.3 Three Non-clique Three-point Lines

We are now interested in the ways that a clique element e can be on three non-clique three-point lines, but be in no $U_{2,4}$ -restrictions. Recall from Section 6.4.2 that there are eleven ways that such a clique element can be on two non-clique three-point lines, displayed in Figure 6.24. As such, if we find an element e such that e is in no $U_{2,4}$ -restrictions and on three non-clique three-point lines, then any two of the non-clique three-point lines must form one of the eleven configurations.

As it will be useful, we construct Table 6.1 showing how the eleven configurations arise. From this table, for example, we can see that in Ξ (Figure 6.24.5), e lies inside a GG configuration, and is also in a red matching.

Note that although Π (Figure 6.24.6) looks like it contains a GG (Figure 6.17.1), it does not. In Π , there are two GR (Figure 6.17.2) configurations, and e is not on a line with the two green triangles.

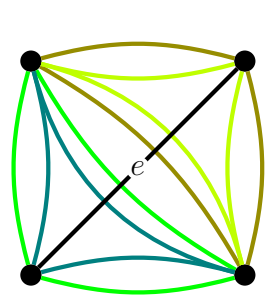
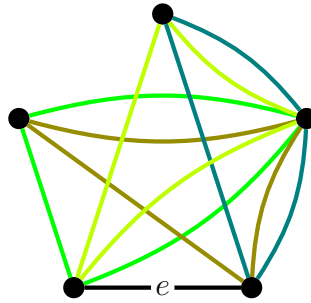
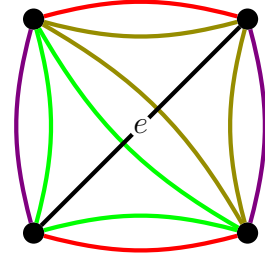
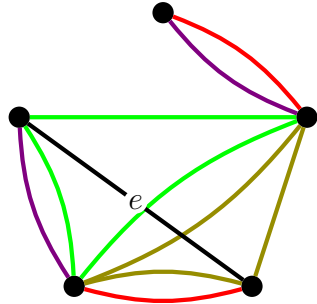
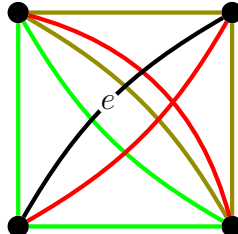
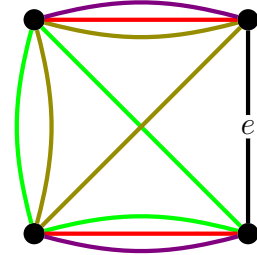
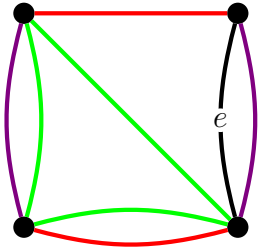
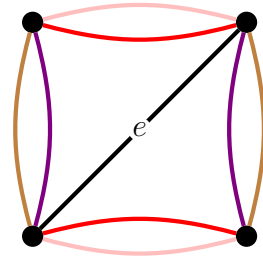
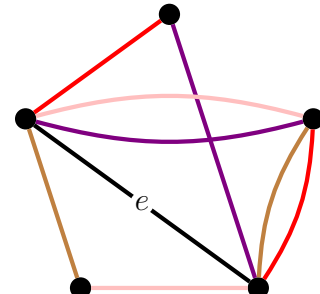
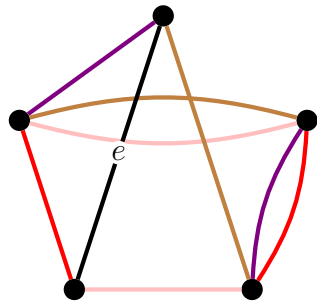
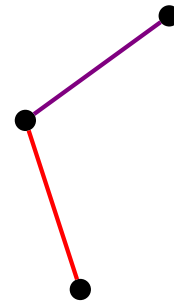
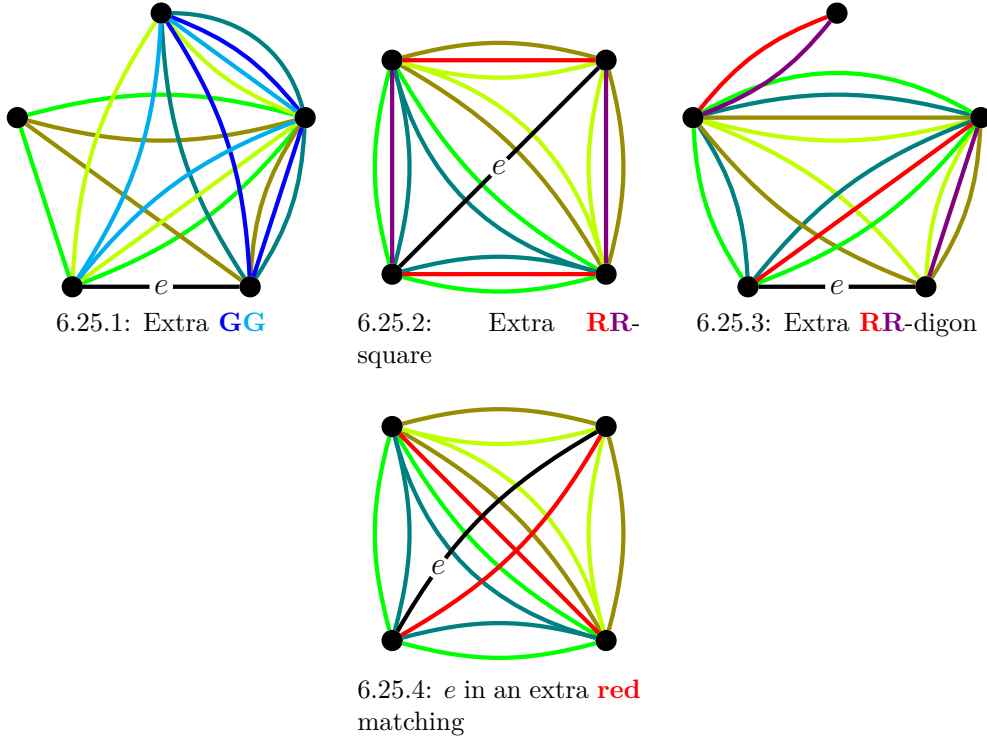
6.24.1: Γ (\mathbf{GG} and \mathbf{GG})6.24.2: Δ (\mathbf{GG} and \mathbf{GG})6.24.3: Θ (\mathbf{GG} and \mathbf{RR} -square)6.24.4: Λ (\mathbf{GG} and \mathbf{RR} -digon)6.24.5: Ξ (\mathbf{GG} and e in a \mathbf{red} matching)6.24.6: Π (\mathbf{GR} and \mathbf{GR})6.24.7: Σ (\mathbf{GR} and e in a \mathbf{red} matching)6.24.8: Υ (\mathbf{RR} -square and \mathbf{RR} -square)6.24.9: Φ (\mathbf{RR} -square and \mathbf{RR} -square)6.24.10: Ψ (\mathbf{RR} -digon and \mathbf{RR} -digon)6.24.11: Ω (e in a \mathbf{red} matching and e in a \mathbf{red} matching)

Figure 6.24: Two Non-clique Three-point Lines

Figure 6.25: Starting from a Γ

6.4.3.1 Γ

We start by considering the case where the first two lines form a Γ (Figure 6.24.1). From Table 6.1, we can see that this particular configuration is constructed from two GG (Figure 6.17.1) three-point lines. Also from Table 6.1, we can instantly dismiss the case where the third non-clique three-point line is a GR (Figure 6.17.2), as there is no way that a GG and a GR can combine to give two non-clique three-point lines.

Identify a GG inside the Γ . We are going to add another GG onto this GG. From Table 6.1, we see that there are two ways to do this, either as a Γ or a Δ (Figure 6.24.2). If we choose a Γ , then we will get a QI (Figure 6.8.1), which is forbidden. Hence we must choose a Δ . It is easier to see this configuration if we start with a Δ and double-up one of the GGs inside it, giving a Γ , as shown in Figure 6.25.1. In this case, `all_lines(5, [[0, 1, 4], [0, 1, 4], [0, 3, 6], [0, 3, 6], [1, 2, 7], [2, 3, 9]], 8)` returns the empty list, so this configuration is forbidden.

The next case is to once again identify a GG inside Γ , and extend this GG into

a Θ (Figure 6.24.3) by adding a RR-square (Figure 6.17.3). This is shown in Figure 6.25.2. In this case, `all_lines(4, [[0, 2, 4], [0, 2, 4], [1, 2, 5], [1, 2, 5], [0, 5], [1, 4]], 3)` returns the empty list, so this configuration is forbidden.

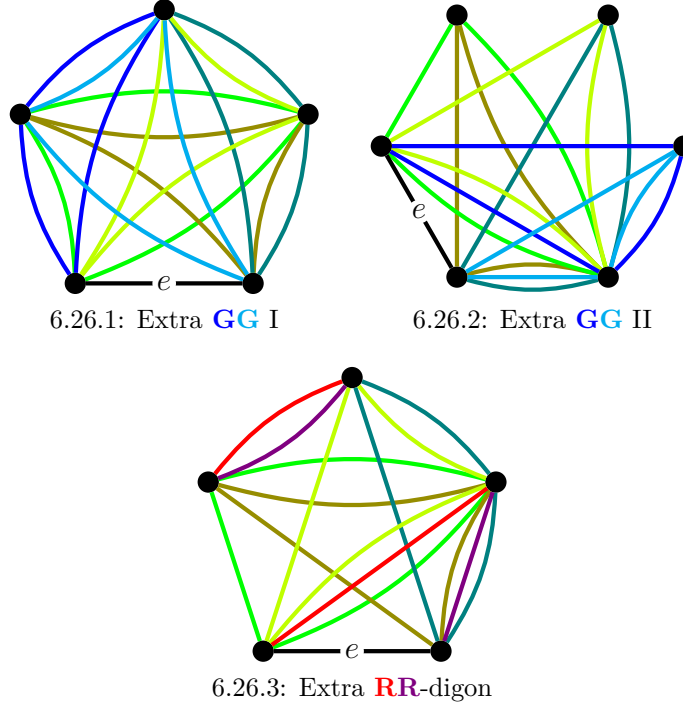
The next case is to once again identify a GG inside Γ , and extend this GG into a Λ (Figure 6.24.4) by adding a RR-digon (Figure 6.17.4). This is shown in Figure 6.25.3. In this case, we can see a CC (Figure 6.12.4), so this configuration is forbidden.

The final case is to identify a GG inside Γ , and extend this GG into a Ξ (Figure 6.24.5) by making e part of a red matching. This is shown in Figure 6.25.4. In this case, `all_lines(4, [[0, 2, 4], [0, 2, 4], [1, 2, 5], [1, 2, 5], [2, 3]], 3)` returns the empty list, hence this configuration is forbidden.

6.4.3.2 Δ

We now consider the case where the first two lines form a Δ (Figure 6.24.1). From Table 6.1, we can see that this configuration is constructed from two GG (Figure 6.17.1) three-point lines. Also from Table 6.1, we can instantly dismiss the case where the third non-clique three-point line is a GR (Figure 6.17.2), as there is no way that a GG and a GR can combine to give two non-clique three-point lines.

Identify a GG inside the Δ . We are going to add another GG onto this GG. From Table 6.1, we see that there are two ways to do this, either as a Γ or a Δ (Figure 6.24.2). If we choose a Γ , then we are in a previous case. Hence we must choose a Δ . If we remain on five vertices, then it is possible to add in an extra GG in a Δ configuration with both existing GGs, as shown in Figure 6.26.1. However, this configuration contains an OM (Figure 6.8.2), and so is forbidden. We now try to add in a third GG while using six vertices. Note that GG is determined by an edge of the complete graph vertex-disjoint from e . This edge must touch the sixth vertex. If this edge does not touch a particular existing green triangle, then we can find a TL (Figure 6.8.3), which is forbidden. Hence this edge must touch all existing green triangles, and there is only one place it could go, shown in Figure 6.26.2. In this case, `all_lines(6, [[0, 1, 5], [0, 2, 6], [0, 4, 8], [1, 3,`

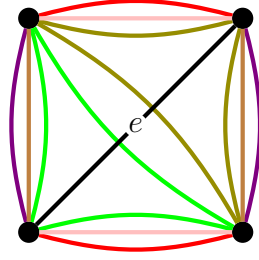
Figure 6.26: Starting from a Δ

10], [2, 3, 12], [3, 4, 14]], 7) outputs that e can be on three non-clique three-point lines. We call this the line-star configuration, shown in Figure 6.34.1.

The next case is to once again identify a GG inside Δ , and extend this GG into a Θ (Figure 6.24.3) by adding a RR-square (Figure 6.17.3). However we place the two red matchings, we are unable to make every red matching edge touch every green triangle, so this case is a contradiction to Lemma 6.3.4.

The next case is to once again identify a GG inside Δ , and extend this GG into a Λ (Figure 6.24.4) by adding a RR-digon (Figure 6.17.4). The path component of the digon must touch the vertex that every existing green triangle is incident with, as otherwise we would have a contradiction to Lemma 6.3.4. The parallel component must also touch all four existing green triangles, and hence there is only one place it could go, regardless of the number of vertices. This configuration, shown in Figure 6.26.3, contains a PM (Figure 6.12.3), and so is forbidden.

The final case is to identify a GG inside Γ , and extend this GG into a Ξ (Figure 6.24.5) by making e part of a red matching. There is only one way to

Figure 6.27: Starting from a Θ

do this, and it fails to turn the other GG into a Ξ , so this case is discarded.

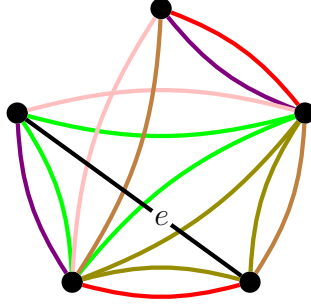
6.4.3.3 Θ

We now consider the case where the first two lines form a Θ (Figure 6.24.3). From Table 6.1, we can see that this configuration is constructed from a GG (Figure 6.17.1) three-point line and a RR-square (Figure 6.17.3) three-point line. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line is a RR-square, as all other possibilities either do not occur or have already been considered. In particular, we do not need to consider adding a GG (Figure 6.17.1), as then the two GGs would have to form either a Γ (Figure 6.24.1) or Δ (Figure 6.24.2), which we have already considered.

From Table 6.1, we see that we can add in a RR-square so that it is either in a Υ (Figure 6.24.8) or a Φ (Figure 6.24.9) with the existing RR-square. However, it is not possible to have a Φ , as this requires five vertices, and such a configuration would not have each of the RR-squares in a θ (Figure 6.24.3) with the existing GG, so cannot occur. Hence there is only one configuration possible, shown in Figure 6.27. In this case, `all_lines(4, [[0, 2, 4], [1, 2, 5], [0, 5], [0, 5], [1, 4], [1, 4]], 3)` outputs the empty list, so this case is forbidden.

6.4.3.4 Λ

We now consider the case where the first two lines form a Λ (Figure 6.24.4). From Table 6.1, we can see that this configuration is constructed from a GG (Figure 6.17.1) three-point line and a RR-digon (Figure 6.17.4) three-point

Figure 6.28: Starting from a Λ

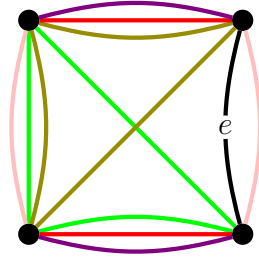
line. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line is a RR-digon, as all other possibilities either do not occur or have already been considered. In particular, we do not need to consider adding a GG (Figure 6.17.1), as then the two GGs would have to form either a Γ (Figure 6.24.1) or Δ (Figure 6.24.2), which we have already considered.

From Table 6.1, we see that we must add on the second RR-digon in a Ψ (Figure 6.24.10) configuration with the existing RR-digon. There is only one way to do this, shown in Figure 6.28. In this case, we can find a HN (Figure 6.13.1), so this case is forbidden.

6.4.3.5 Ξ

We now consider the case where the first two lines form a Ξ (Figure 6.24.5). From Table 6.1, we can see that this configuration is constructed from a GG (Figure 6.17.1) three-point line with the second three-point line coming from e being in a red matching. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line arises from e being on a red matching, as all other possibilities either do not occur or have already been considered. Note that we cannot combine a red matching containing e with a GR, as then the GR and the GG would have to combine to form two non-clique three-point lines containing e , which by Table 6.1 we see does not happen.

From Table 6.1, we see that we put e into a second red matching in a Ω (Figure 6.24.11) configuration with the existing red matching containing e . This requires five vertices, and such a configuration would not have each of

Figure 6.29: Starting from a Π

the red matchings containing e in a Ξ configuration with the GG, and so this cannot occur.

6.4.3.6 Π

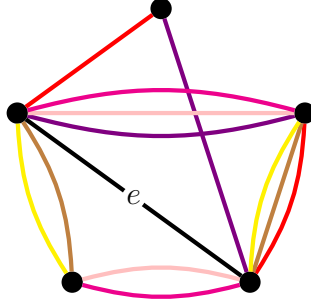
We now consider the case where the first two lines form a Π (Figure 6.24.6). From Table 6.1, we can see that this configuration is constructed from two GR (Figure 6.17.2) three-point lines. Also from Table 6.1, we note that the only cases we need to consider are the case where the third non-clique three-point line arises from a GR, and the case where the third non-clique three-point line arises from e being in a red matching as all other possibilities do not occur.

Identify a GR inside the Π . We are going to add another GR onto this GR. From Table 6.1, we see that there is one way to do this, as a Π . However, this will not add the third GR into a Π with the second GR, as is required. Hence this case is forbidden.

The final case is to identify a GR inside Π , and extend this GR into a Σ (Figure 6.24.7) by making e part of a red matching. This is shown in Figure 6.29. In this case, `all_lines(4, [[0, 1, 3], [0, 2, 4], [0, 5], [1, 4], [1, 4]], 5)` outputs that e is on three non-clique three-point lines. We call this the Betsy Ross configuration, shown in Figure 6.34.2.

6.4.3.7 Σ

We now consider the case where the first two lines form a Σ (Figure 6.24.7). From Table 6.1, we can see that this particular configuration is constructed from a GR (Figure 6.17.2) three-point line with the second non-clique three-

Figure 6.30: Starting from a Υ

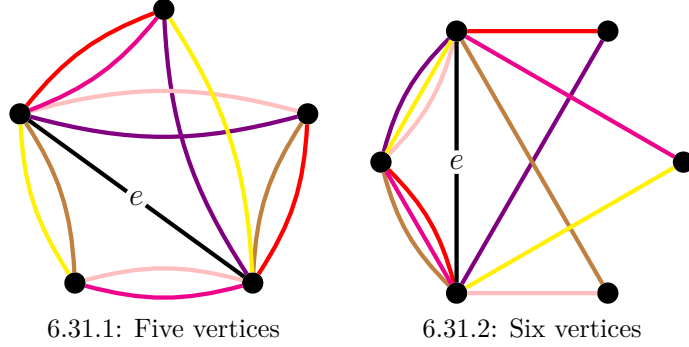
point line coming from e being in a red matching. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line arises from e being in a red matching as all other possibilities either do not occur or have already been considered.

From Table 6.1, we see that we put e into a second red matching in a Ω (Figure 6.24.11) configuration with the existing red matching containing e . This requires five vertices, and such a configuration would not have each of the red matchings containing e in a Σ configuration with the GR, and so this cannot occur.

6.4.3.8 Υ

We now consider the case where the first two lines form a Υ (Figure 6.24.8). From Table 6.1, we can see that this particular configuration is constructed from two RR-square (Figure 6.17.3) three-point lines. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line arises from a RR-square as all other possibilities either do not occur or have already been considered.

Identify a RR-square inside the Υ . We are going to add another RR-square onto this RR-square. From Table 6.1, we see that there are two ways to do this, either as a Υ or a Φ (Figure 6.24.9). If we choose a Υ , then we will get a TM (Figure 6.10.2), which is forbidden. Hence we must choose a Φ . It is easier to see this configuration if we start with a Φ and double-up one of the RR-squares inside it, giving a Υ , as shown in Figure 6.30. In this case, `all_lines(5, [[1, 5], [1, 9], [1, 9], [2, 4], [2, 8], [2, 8]], 7)` returns the empty list, so this configuration is forbidden.

Figure 6.31: Starting from a Φ

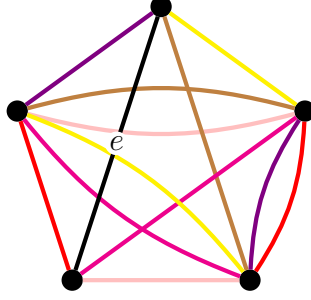
6.4.3.9 Φ

We now consider the case where the first two lines form a Φ (Figure 6.24.9). From Table 6.1, we can see that this particular configuration is constructed from two RR-square (Figure 6.17.3) three-point lines. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line arises from a RR-square as all other possibilities either do not occur or have already been considered.

Furthermore, we see from Table 6.1 that there are two ways to add in the third RR-square relative to each existing RR-square: Υ (Figure 6.24.8), which has already been covered, and Φ . So any two RR-squares in our final configuration must form a Φ .

We start by attempting to create such a configuration on five vertices. There are three choices for a vertex for our third RR-square to avoid, and two of these vertices would lead to a Υ . Hence there is only one place for the third RR-square, shown in Figure 6.31.1. In this case, we can find a BC (Figure 6.10.3), and therefore this case is forbidden.

We now try to construct such a configuration on six vertices. In Φ , the two RR-squares have two edges that are adjacent in both RR-squares. As such, there is only one place that we can put the third RR-square without ruining this condition for one of the existing RR-squares. This configuration is shown in Figure 6.31.2. In this case, `all_lines(6, [[0, 14], [4, 7], [5, 14], [6, 14], [7, 11], [7, 13]], 8)` outputs that e is on three non-clique three-point lines. We call this the two-parallel configuration, shown in Figure 6.34.3.

Figure 6.32: Starting from a Ψ

6.4.3.10 Ψ

We now consider the case where the first two lines form a Ψ (Figure 6.24.10). From Table 6.1, we can see that this particular configuration is constructed from two RR-digon (Figure 6.17.4) three-point lines. Also from Table 6.1, we note that the only case we need to consider is the case where the third non-clique three-point line arises from a RR-digon as all other possibilities either do not occur or have already been considered.

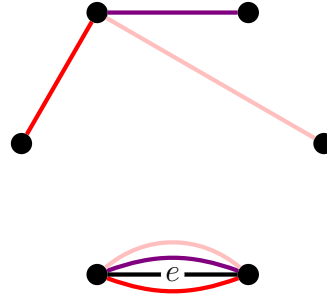
We also note from Table 6.1 that there is only way to add in the third RR-digon, namely that any two of the RR-digons must form a Ψ . In light of this, there is only one way to add in the third RR-digon, shown in Figure 6.32. In this case, we can find a XS (Figure 6.10.1), so this case is forbidden.

6.4.3.11 Ω

Finally, we are in the case where the first two lines form a Ω . From Table 6.1, we can see that this configuration is formed by having e on two red matchings. Also, we note that the third non-clique three-point line must also be made by having e on a red matching, as all other possibilities either do not occur or have already been considered. Furthermore, any two red matchings in our final configuration must form a Ω .

We start by constructing this on five vertices. Taking into account the previously mentioned restriction, there is only one place that we could place the third red matching, which is in such a way that an AE (Figure 6.11.2) configuration results, which is forbidden.

We now construct this on six vertices. Taking into account the previously

Figure 6.33: Starting from a Ω

mentioned restriction, there is only one place that we could place the third red matching, shown in Figure 6.33. In this case, `all_lines(6, [[0, 11], [0, 13], [0, 14]], 0)` outputs that e is on three non-clique three-point lines. We call this the matching-star configuration, shown in Figure 6.34.4.

6.4.3.12 Summary

To summarise, there are four ways to have a clique element e on three non-clique three-point lines, yet no $U_{2,4}$ -restrictions. These are shown in Figure 6.34.

The line-star configuration consists of a highlighted edge e and a star that is vertex-disjoint from e . Each edge of the star is contained in two green triangles, one using each of the end vertices of e . A schematic is given in Figure 6.44.

The Betsy Ross configuration consists of two green triangles and three red triangles on K_4 . If we label the vertices of K_4 such that e is the line ab , then the two green triangles are acd and abd , while the three red matchings are (ac, bd) twice and (ab, cd) .

The two-parallel configuration consists of three distinguished vertices, two of which are the ends of e , and a collection of other vertices. For each non-distinguished vertex, a RR-square not containing e is formed that uses the three distinguished vertices.

The matching-star configuration consists of a distinguished edge e and a star that is vertex-disjoint from e . For each edge of this star, form a matching that uses this edge and e .

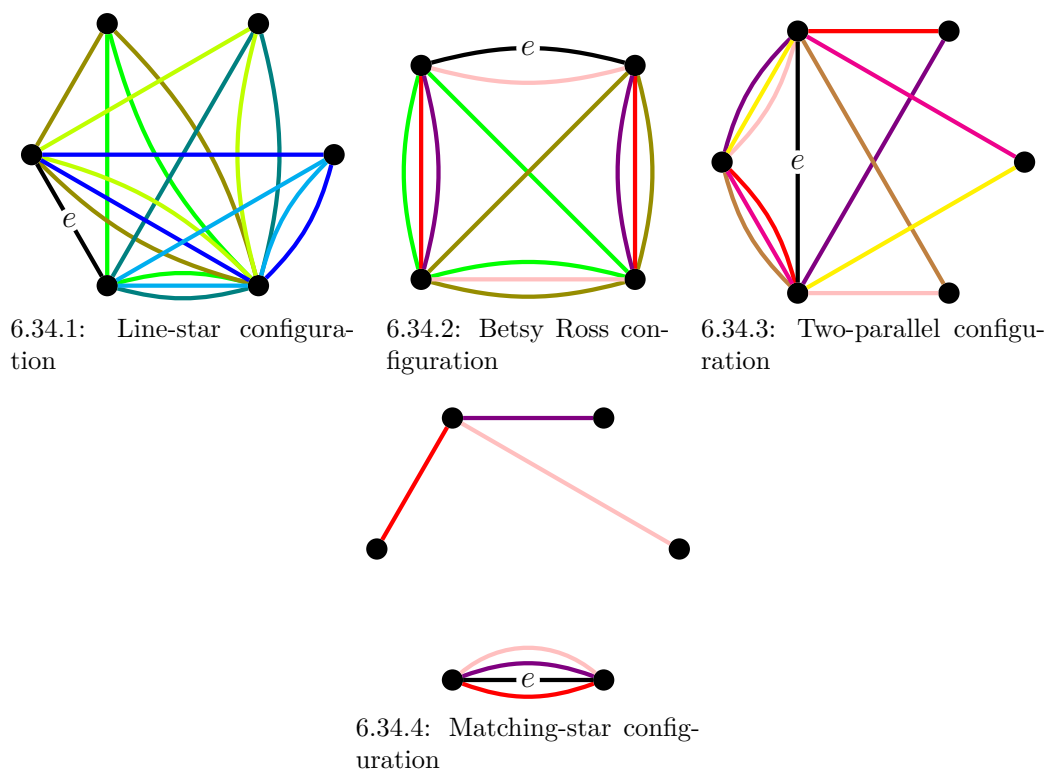


Figure 6.34: Three non-clique three-point lines

6.5 $U_{2,4}$ -restrictions

We now analyse how an augmented clique can give rise to an $U_{2,4}$ -restriction containing at least one element of the clique. Hence an $U_{2,4}$ -restriction must contain one, two, or three non-clique elements. We consider each case in turn.

6.5.1 One Non-clique Element

We consider the case that a $U_{2,4}$ -restriction contains exactly one non-clique element. In this case, the $U_{2,4}$ -restriction contains a triangle of the clique. There are two ways to add in one non-clique element: a red matching or a green triangle. The red matching does not create any $U_{2,4}$ -restrictions, while the green triangle creates one.

6.5.2 Two Non-clique Elements

We consider the case that a $U_{2,4}$ -restriction contains exactly two non-clique elements. If we have a $U_{2,4}$ -restriction containing exactly two non-clique elements, then we can find a $U_{2,3}$ -restriction containing one clique element and two non-clique elements. In Sections 6.4.1.2 – 6.4.1.4, we looked for long lines that contained two non-clique elements. The result of this analysis was that the only way to obtain a $U_{2,4}$ -restriction with exactly two non-clique elements is the doubled-up red matching (Figure 6.15.2).

6.5.3 Three Non-clique Elements

We consider the case that a $U_{2,4}$ -restriction contains exactly three non-clique elements. There are four different cases to consider, and we go through them in a methodical way.

6.5.3.1 Three Green Triangles

From Section 6.4.1.2, we know that the first two green triangles meet at a triangle, an edge, or a vertex, so we only need to consider how to place the third triangle.

We start with the first two triangles meeting at a triangle (Figure 6.14.1). We cannot use only these three vertices, as that would create a QI (Figure 6.8.1), which is forbidden. If we use one extra vertex, the only way to place the third green triangle is as in GGG_1 (Figure 6.35.1). If we use two extra vertices, we get GGG_2 (Figure 6.35.2). We cannot use any more vertices, as this would lead to vertex-disjoint green triangles, which is forbidden by TL (Figure 6.8.3).

Now we start with the first two triangles meeting at an edge (Figure 6.14.2). When we add in the third green triangle, we cannot add in it on top of an existing triangle, as this case has already been considered. With this in mind, there is only one way to place the third green triangle if we use only the existing four vertices, GGG_3 (Figure 6.35.3).

With five vertices, the third green triangle is determined by an edge in the span of the first two triangles. There are three such edges – the edge that

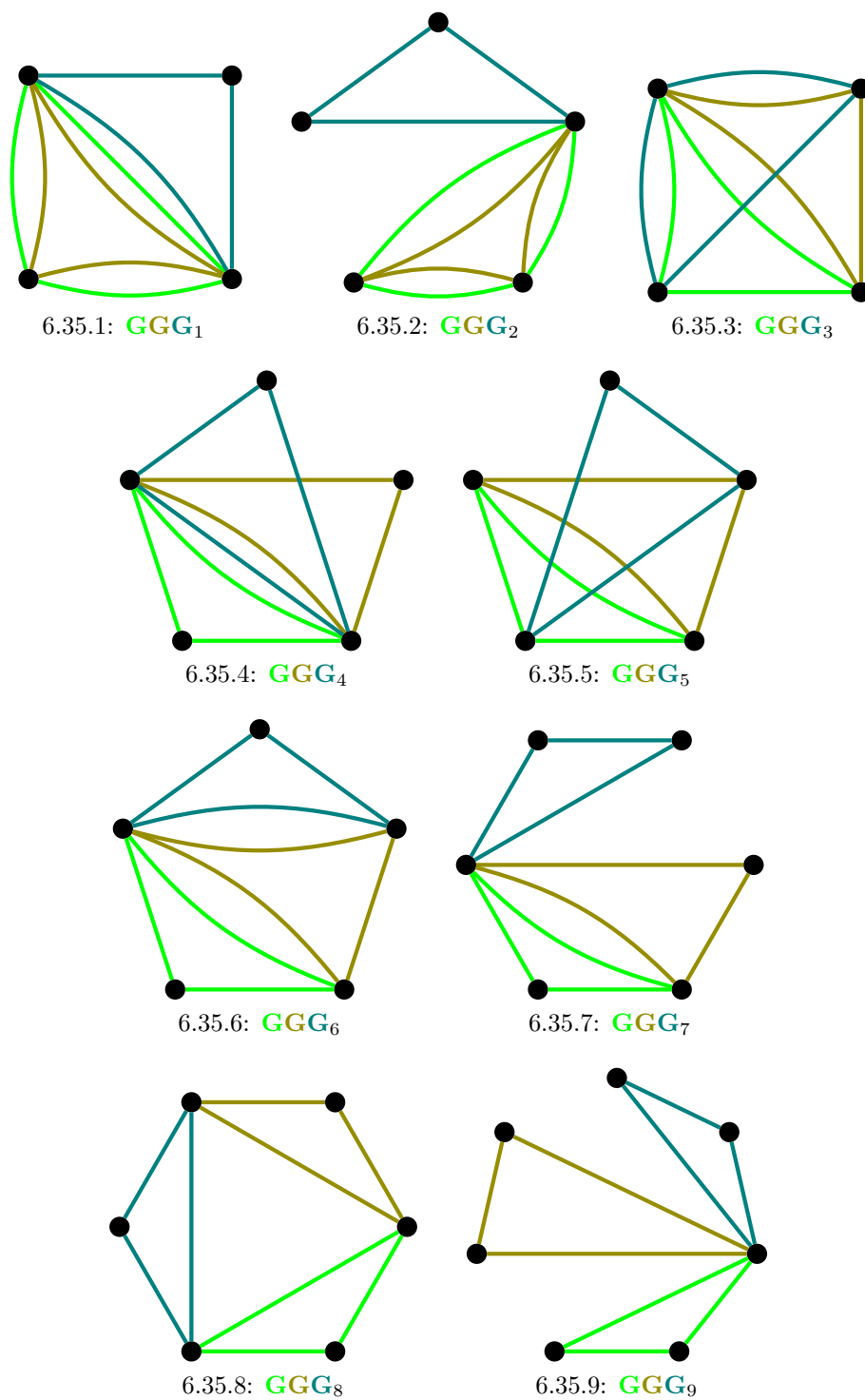


Figure 6.35: Three Green Triangles

is in both existing triangles (GGG_4 , Figure 6.35.4), the edge that is in no existing triangles (GGG_5 , Figure 6.35.5), and the edge that is in exactly one of the existing triangles (GGG_6 , Figure 6.35.6).

With six vertices, the third triangle is determined by an existing vertex. This existing vertex must touch both existing triangles (because of TL, Figure 6.8.3), so there is only one place to put it, giving GGG_7 (Figure 6.35.7).

Finally, we start with the first two triangles meeting at a vertex (Figure 6.14.3). The final green triangle must meet both existing triangles at a vertex, and not use any existing edges, or we are in a previously considered case. There is no way to do this with no extra vertices. With one extra vertex, there is only one way, GGG_8 (Figure 6.35.8). With two extra vertices, there is also only one to do this, GGG_9 (Figure 6.35.9).

We now go through each configuration in turn and check for the existence of $U_{2,4}$ -restrictions containing all three of the extension elements. These checks are done by the `lines()` function from Appendix 6.A.4. The function calls, and an interpretation of the output are given below.

GGG_1 `lines(4, [[0, 2, 4], [0, 2, 4], [1, 2, 5]])` outputs that there is no line containing the three extension elements and a clique element.

GGG_2 `lines(5, [[0, 2, 5], [1, 3, 8], [1, 3, 8]])` outputs that there is no line containing the three extension elements and a clique element.

GGG_3 `lines(4, [[0, 1, 3], [0, 2, 4], [1, 2, 5]])` outputs that there is no line containing the three extension elements and a clique element.

GGG_4 `lines(5, [[1, 2, 7], [4, 5, 7], [7, 8, 9]])` outputs that there is no line containing the three extension elements and a clique element.

GGG_5 This is OM (Figure 6.8.2), and so is forbidden.

GGG_6 `lines(5, [[0, 2, 5], [1, 2, 7], [7, 8, 9]])` outputs that there is no line containing the three extension elements and a clique element.

GGG_7 `lines(6, [[0, 3, 7], [1, 3, 10], [12, 13, 14]])` outputs that there is no line containing the three extension elements and a

clique element.

GGG₈ This is TB (Figure 6.8.4), and so is forbidden.

GGG₉ This is RA (Figure 6.8.6), and so is forbidden.

So there are no configurations of interest.

6.5.3.2 Two Green Triangles and One Red Matching

From Section 6.4.1.2, we know that the first two green triangles meet at a triangle, an edge, or a vertex, so we only need to consider how to place the matching.

First note that we cannot add more than two extra vertices, as if we added three extra vertices, then one edge of the matching must be vertex-disjoint from the existing green triangles, contradicting Lemma 6.3.4.

We start with the two green triangles meeting at a triangle (Figure 6.14.1). We cannot use three vertices, as a red matching requires four vertices. With one extra vertex, the only thing we can do is GGR₁ (Figure 6.36.1). With two extra vertices, we cannot have a matching edge go between these vertices, as that would contradict Lemma 6.3.4. This leaves only GGR₂ (Figure 6.36.2).

Now we start with the two green triangles meeting at an edge (Figure 6.14.2). If we stay on four vertices, there are two places where a red matching could go – either both edges in parallel with an existing triangle edge (GGR₃, Figure 6.36.3), or using the edge that the existing triangles avoid (GGR₄, Figure 6.36.4). We now add in one extra vertex. Due to Lemma 6.3.4, the first edge of the matching must go to one of the vertices that both existing green triangles touch. This leaves two places that the final matching edge can go – either in parallel with one of the triangle edges (GGR₅, Figure 6.36.5), or incident with the two vertices that are incident with exactly one green triangle (GGR₆, Figure 6.36.6). With two extra vertices, the restrictions already discussed only allow one configuration, GGR₇ (Figure 6.36.7).

Finally, we start with the two green triangles meeting at a vertex (Figure 6.14.3). Starting with the existing five vertices, the first edge placed of the matching could either be in parallel with a triangle edge, or it could not. If it is in parallel, and taking Lemma 6.3.4 into consideration, there is only one place for it to go, namely it must be incident with both green

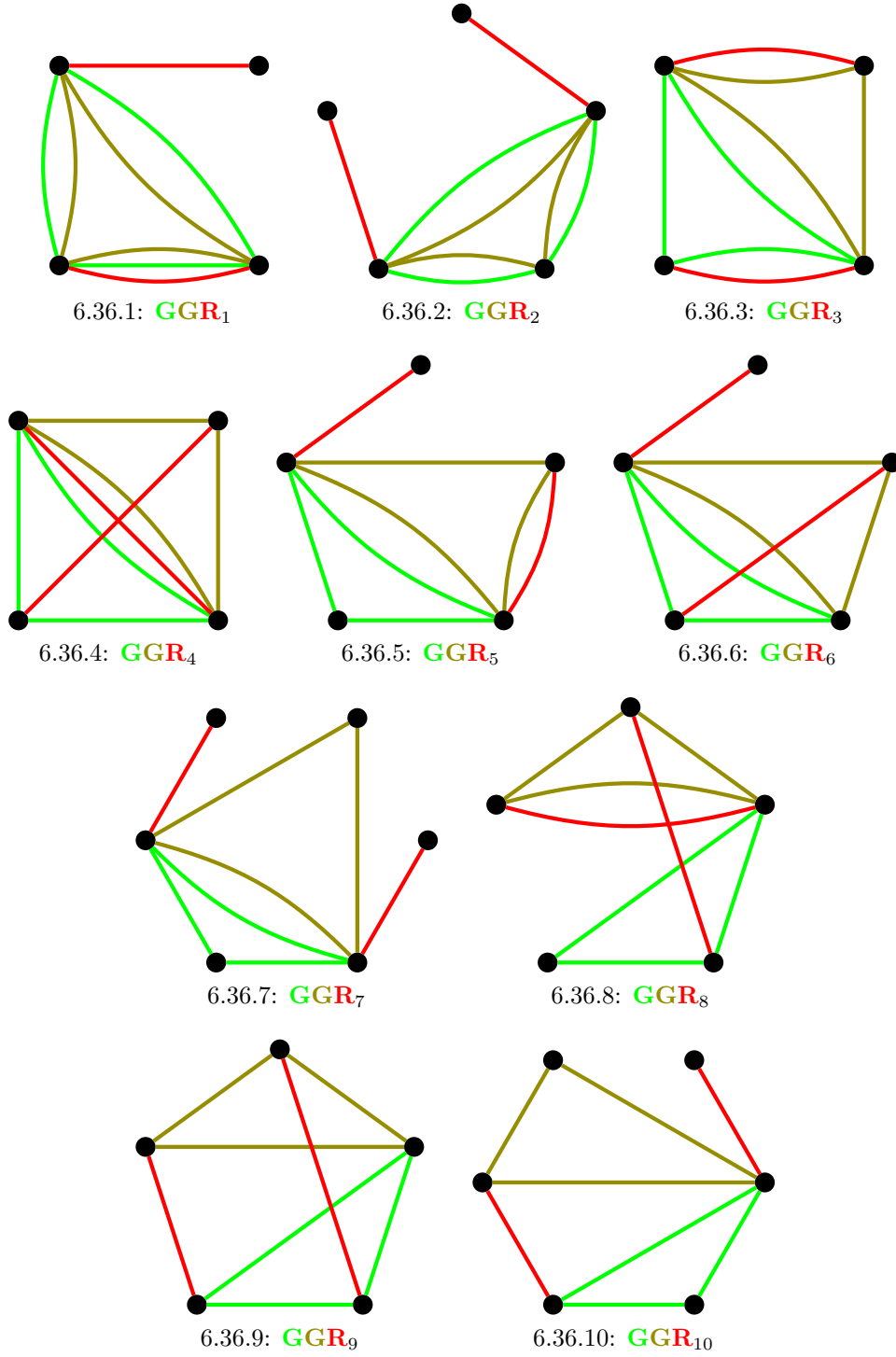


Figure 6.36: Two Green Triangles and One Red Matching

triangles. Taking Lemma 6.3.4 into consideration again, there is also only one place for the second matching edge to go, giving GGR_8 (Figure 6.36.8). If neither edge is in parallel, there are only four vertices that our matching could span, forcing us to have GGR_9 (Figure 6.36.9).

Adding in extra vertices, as every matching edge that uses these new vertices must touch both triangles, and there is only one vertex that both triangles touch, we can only add one extra vertex. Once this vertex, and the matching edge required by it, are added, there is only one place the last matching edge could go, giving GGR_{10} (Figure 6.36.10).

We now go through each configuration in turn and check for the existence of $U_{2,4}$ -restrictions containing all three of the extension elements. These checks are done by the `lines()` function from Appendix 6.A.4. The function calls, and an interpretation of the output are given below.

GGR_1 `lines(4, [[0, 2, 4], [0, 2, 4], [1, 4]])` outputs that there is no line containing the three extension elements and a clique element.

GGR_2 This is CC (Figure 6.12.4), and so is forbidden.

GGR_3 `lines(4, [[0, 2, 4], [1, 2, 5], [1, 4]])` outputs that $[a, b, c, 3]$ is a line containing the three extension elements. This is \aleph , shown in Figure 6.43.1.

GGR_4 `lines(4, [[0, 2, 4], [1, 2, 5], [2, 3]])` outputs that there is no line containing the three extension elements and a clique element.

GGR_5 `lines(5, [[1, 2, 7], [7, 8, 9], [1, 5]])` outputs that there is no line containing the three extension elements and a clique element.

GGR_6 This is KF (Figure 6.12.2), and so is forbidden.

GGR_7 `lines(6, [[0, 3, 7], [2, 3, 12], [1, 14]])` outputs that there is no line containing the three extension elements and a clique element.

GGR_8 This is PM (Figure 6.12.3), and so is forbidden.

GGR_9 `lines(5, [[0, 2, 5], [1, 3, 8], [4, 9]])` outputs that there is no line containing the three extension elements and a clique element.

GGR_{10} This is SR (Figure 6.13.5), and so is forbidden.

So GGR_3 is the only configuration of interest.

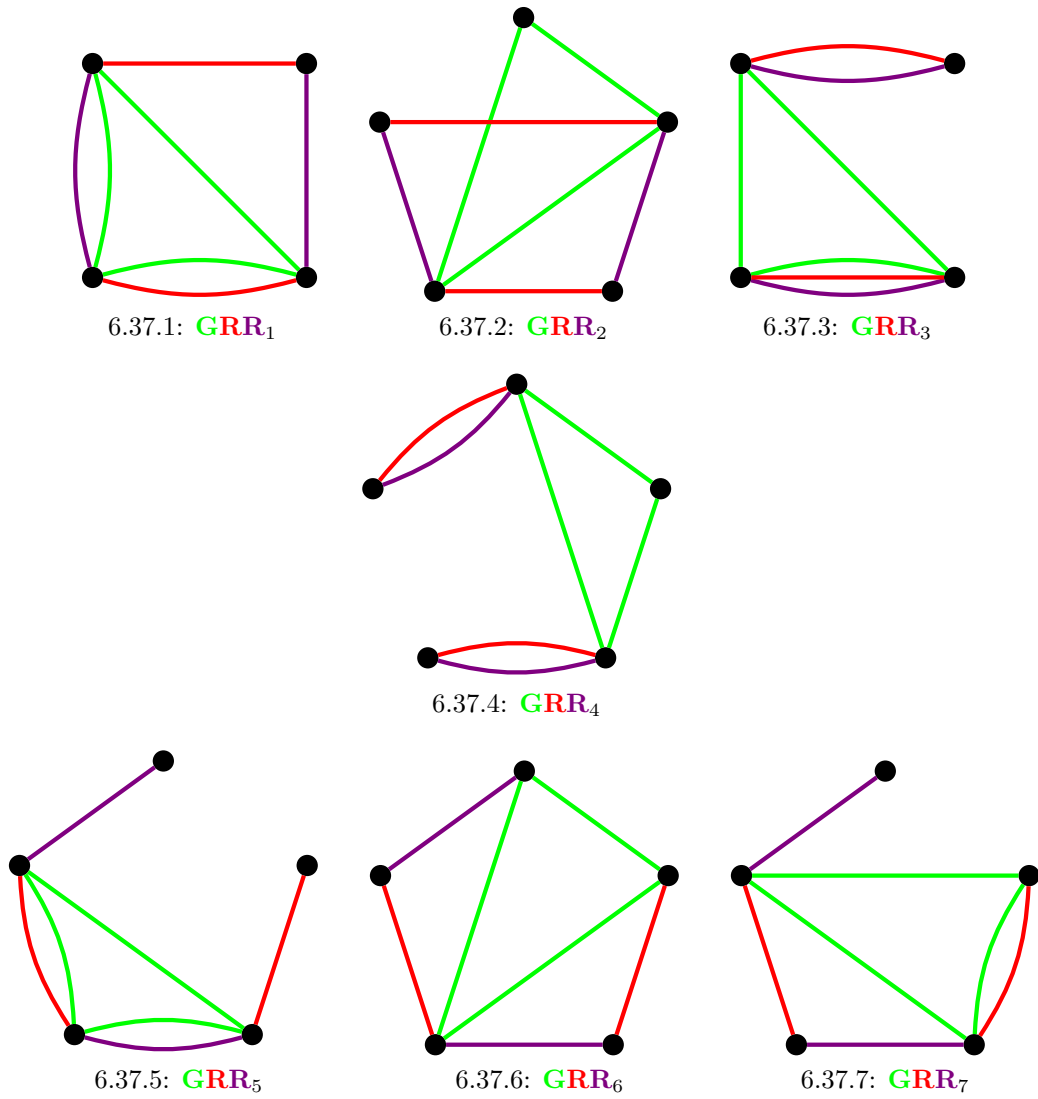


Figure 6.37: One Green Triangle and Two Red Matchings I

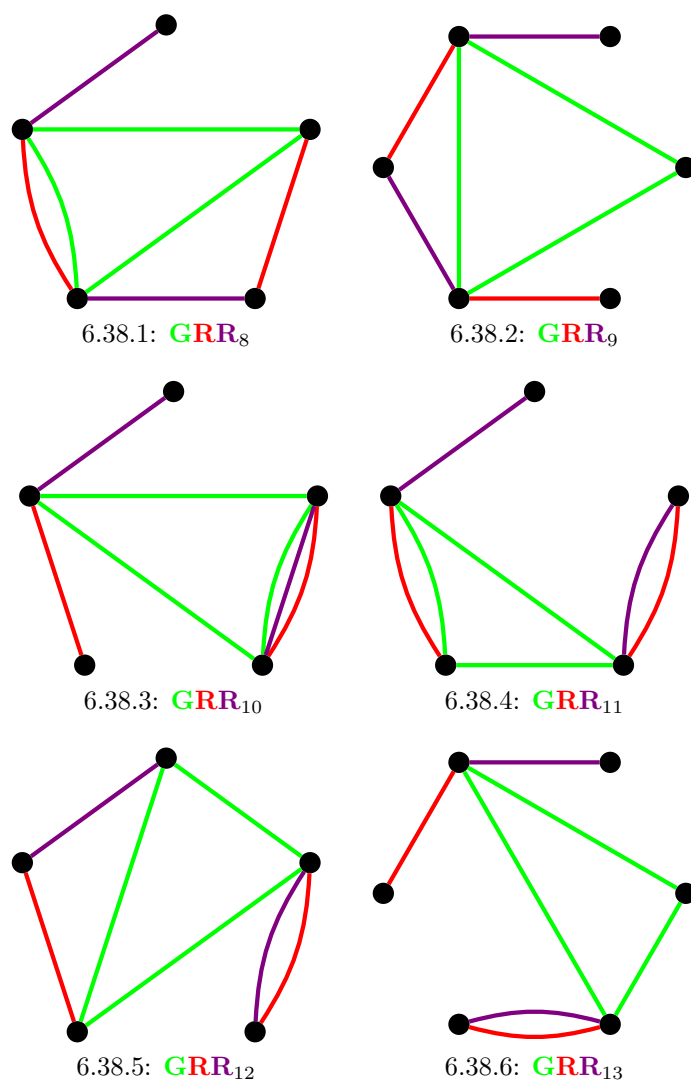


Figure 6.38: One Green Triangle and Two Red Matchings II

6.5.3.3 One Green Triangle and Two Red Matchings

From Section 6.4.1.3 we know that the first two red matchings must form a square, a double, a join, or a parallel. We place the green triangle on each of these in all possible ways.

First note that if we were to add in two additional vertices, we would need to find a vertex that all four matching edges touch because of Lemma 6.3.4. This can never happen, so in this entire case, we can never add more than one vertex.

When we start with the square (Figure 6.15.1), we place the green triangle without adding any additional vertices. There is only one way to do this, GRR_1 (Figure 6.37.1). With one additional vertex, the edge that defines the green triangle must touch all four existing matching edges because of Lemma 6.3.4, and GRR_2 (Figure 6.37.2) is the only way to do this.

Our next case is to start with the double (Figure 6.15.2). First, we place the green triangle without any additional vertices. There is only one way to do this, GRR_3 (Figure 6.37.3). With one additional vertex, the edge that defines the green triangle must touch all four existing matching edges because of Lemma 6.3.4, and GRR_4 (Figure 6.37.4) is the only way to do this.

We now start with the join (Figure 6.15.3) and try to add in a green triangle while remaining on five vertices. To do this, we identify an edge in the span of the two existing matchings and place the triangle disjoint from this edge. The edge cannot be parallel to a matching edge, as then that matching edge would be disjoint from the green triangle, a contradiction to Lemma 6.3.4. This leaves four places for the avoided edge. First, we could pick the edge that would turn the path of matchings into a cycle, this gives us GRR_5 (Figure 6.37.5). Second, we could pick the edge that does not touch either end of the path of matchings, giving us GRR_6 (Figure 6.37.6). Third, we could pick the edge that goes from one end of the path of matchings to the centre of the path, giving us GRR_7 (Figure 6.37.7). Finally, we could pick the edge that goes from one end of the path of matchings to the fourth vertex in the path of matchings, giving GRR_8 (Figure 6.38.1). We now add in one extra vertex. Because of Lemma 6.3.4, the edge that determines the triangle must touch all four existing matching edges. There is only one edge that does this, giving us GRR_9 (Figure 6.38.2).

Finally, we start with the parallel (Figure 6.15.4) and try to add in a green triangle while remaining on five vertices. As before, we identify an edge for our green triangle to avoid. This edge cannot be parallel to any existing matching edge, leaving three choices. First, we could pick the edge that joins the ends of the path, giving GRR_{10} (Figure 6.38.3). Next, we could pick an edge from the parallel class to an end of the path, giving GRR_{11} (Figure 6.38.4). Finally, we could select an edge from the parallel class to the centre of the path, giving GRR_{12} (Figure 6.38.5). We now add in one extra vertex. Because of Lemma 6.3.4, the edge that determines the triangle must touch all four existing matching edges. There is only one edge that does this, giving us GRR_{13} (Figure 6.38.6).

We now go through each configuration in turn and check for the existence of $U_{2,4}$ -restrictions containing all three of the extension elements. These checks are done by the `lines()` function from Appendix 6.A.4. The function calls, and an interpretation of the output are given below.

GRR_1 `lines(4, [[0, 2, 4], [0, 5], [1, 4]])` outputs that $[a, b, c, 3]$ is a line containing the three extension elements. This is \sqsupset , shown in Figure 6.43.2.

GRR_2 `lines(5, [[0, 3, 6], [1, 9], [2, 8]])` outputs that there is no line containing the three extension elements and a clique element.

GRR_3 `lines(4, [[0, 2, 4], [1, 4], [1, 4]])` outputs that there is no line containing the three extension elements and a clique element.

GRR_4 This is CF (Figure 6.12.5), and so is forbidden.

GRR_5 `lines(5, [[7, 8, 9], [1, 9], [5, 8]])` outputs that there is no line containing the three extension elements and a clique element.

GRR_6 This is FS (Figure 6.13.2), and so is forbidden.

GRR_7 This is HN (Figure 6.13.1), and so is forbidden.

GRR_8 This is HM (Figure 6.13.4), and so is forbidden.

GRR_9 This is BP (Figure 6.13.6), and so is forbidden.

GRR_{10} `lines(5, [[1, 2, 7], [1, 5], [1, 9]])` outputs that there is no line containing the three extension elements and a clique element.

GRR_{11} `lines(5, [[7, 8, 9], [1, 5], [1, 9]])` outputs that there is no line containing the three extension elements and a clique element.

GRR_{12} This is WB (Figure 6.13.3), and so is forbidden.

GRR_{13} `lines(6, [[1, 4, 11], [0, 13], [0, 14]])` outputs that there

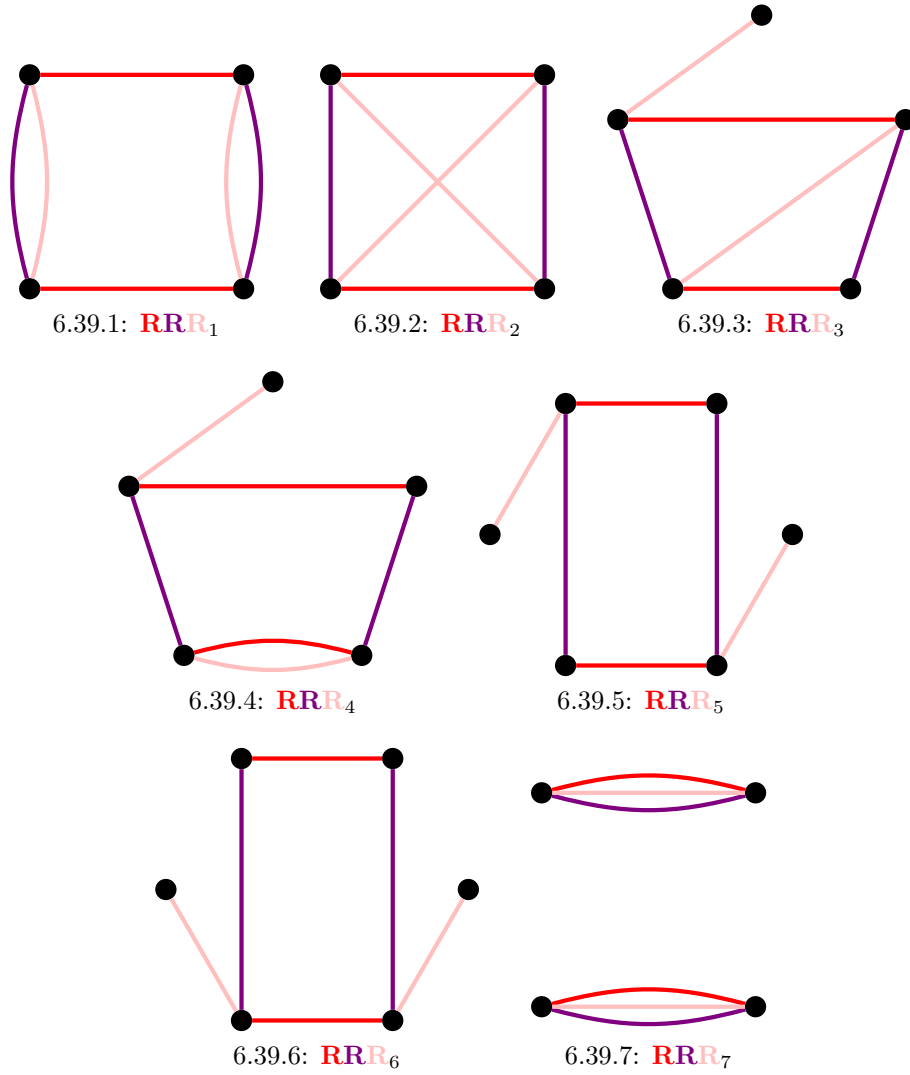


Figure 6.39: Three Red Matchings I

is no line containing the three extension elements and a clique element. So \mathbf{GRR}_1 is the only configuration of interest.

6.5.3.4 Three Red Matchings

From Section 6.4.1.3 we know that the first two red matchings must form a square, a double, a join, or a parallel. We systematically add in the third matching in all possible ways.

First note that we cannot add in more than two new vertices, as adding in three new vertices would force one red matching edge to be vertex-disjoint

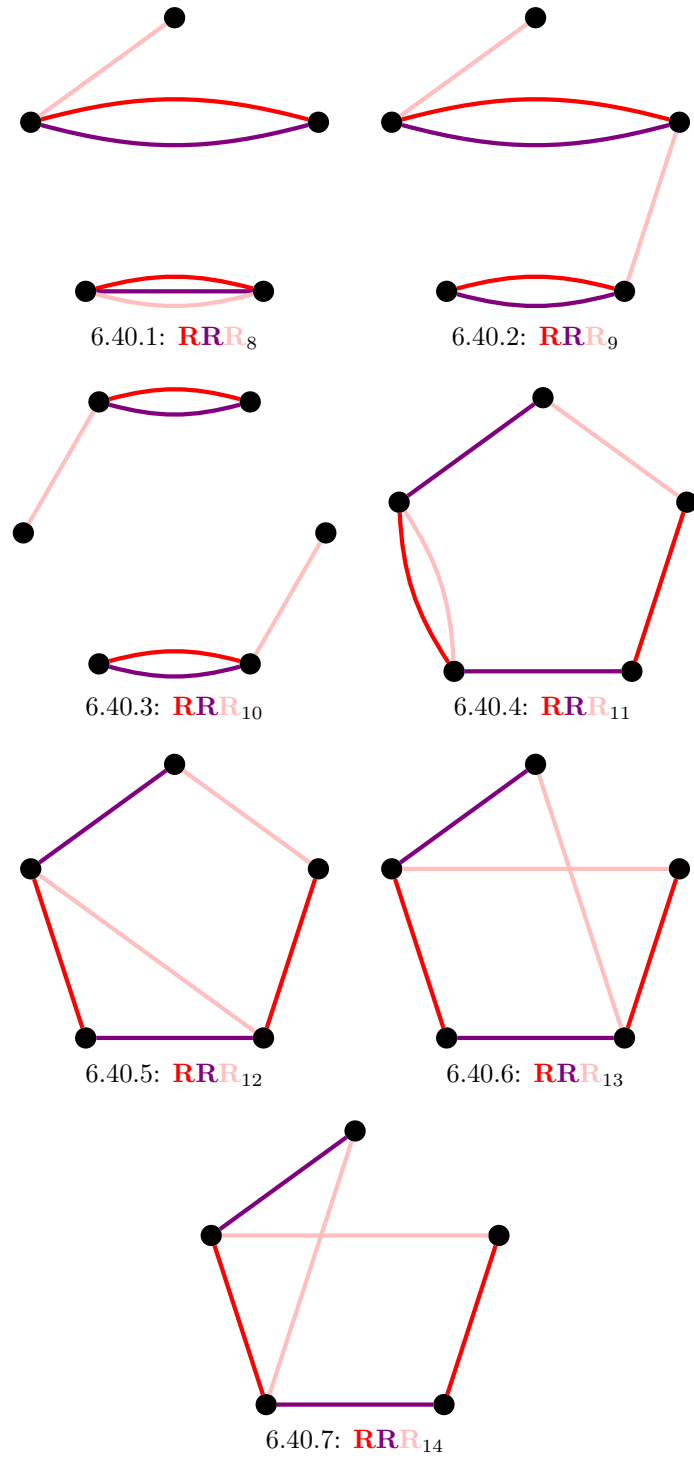


Figure 6.40: Three Red Matchings II

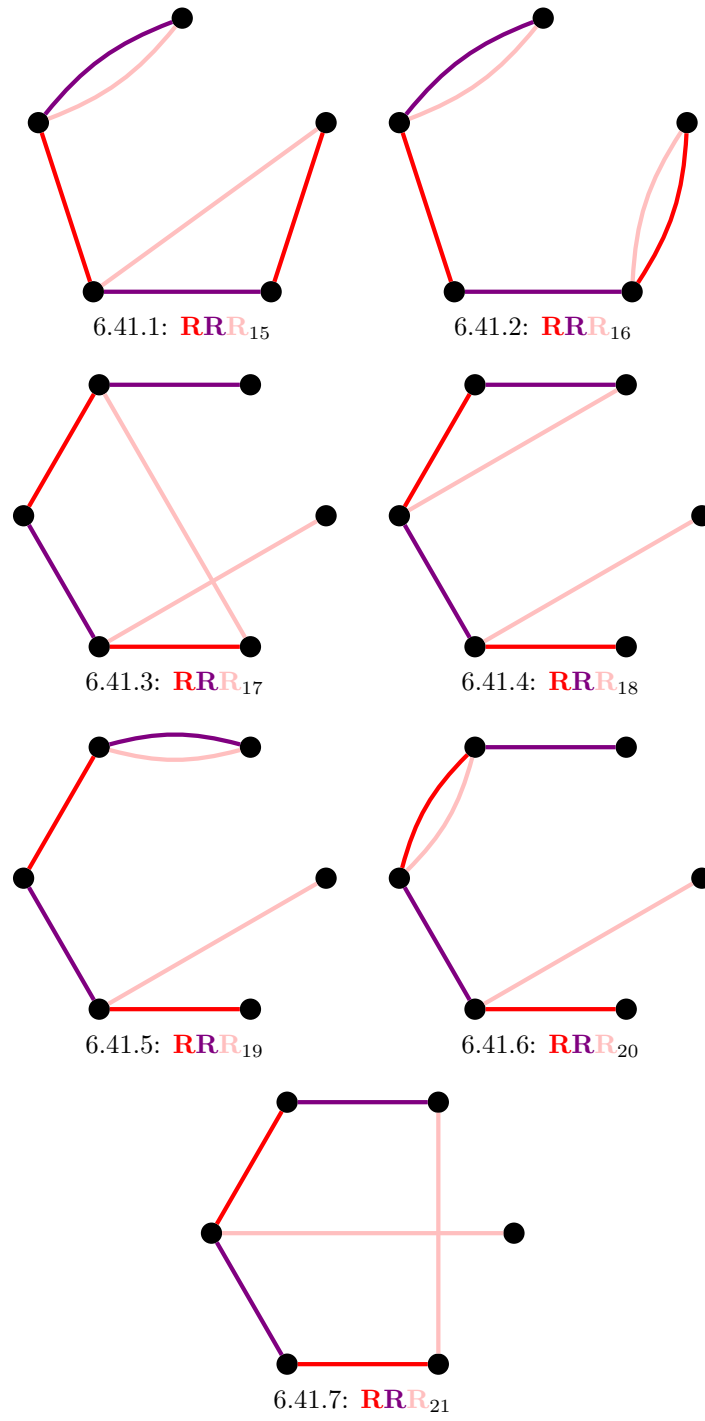


Figure 6.41: Three Red Matchings III

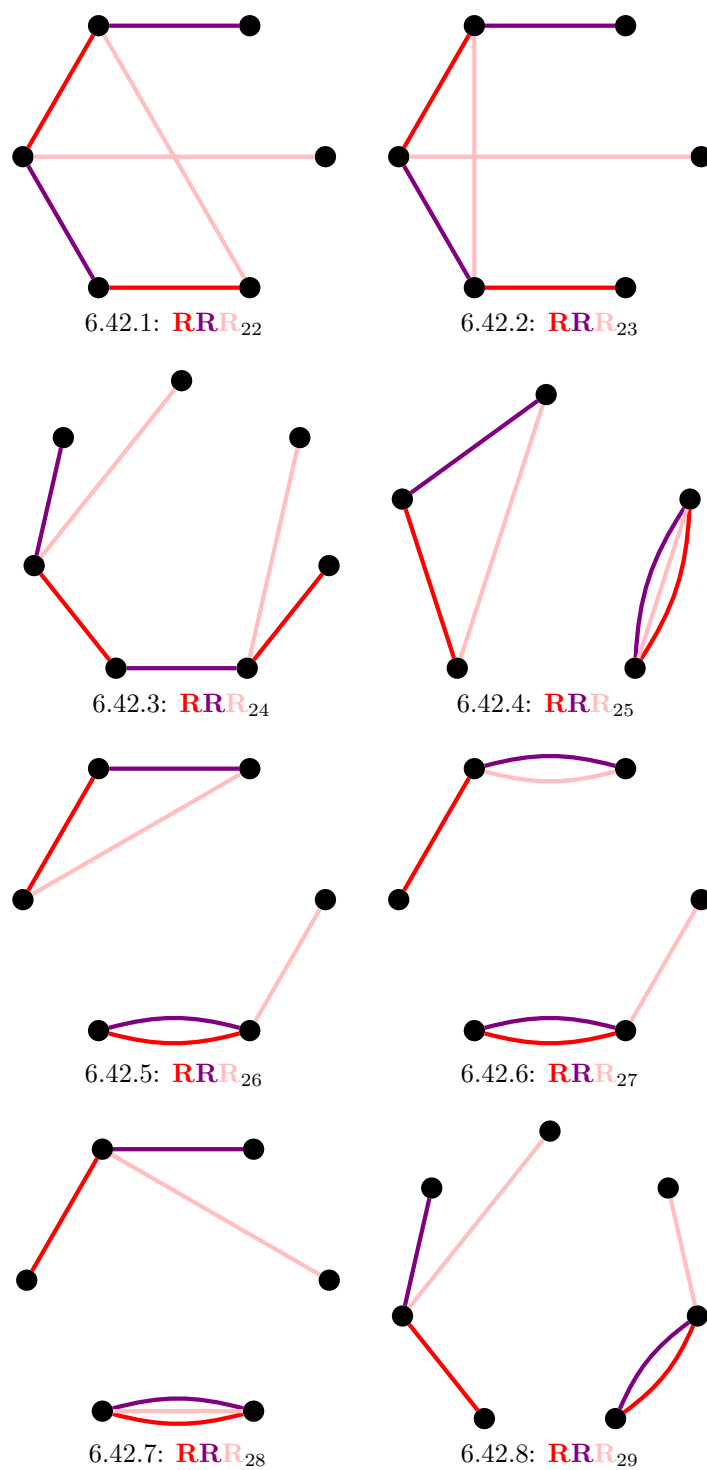


Figure 6.42: Three Red Matchings IV

from all the other matchings, creating a contradiction to Lemma 6.3.5.

Our first case is starting from the square (Figure 6.15.1) and remaining on four vertices. We can either add our third matching in parallel to one of the existing matchings, giving RRR_1 (Figure 6.39.1), or we can add our third matching so that it is edge-disjoint from both existing matchings, giving RRR_2 (Figure 6.39.2).

Our next case is again starting from the square, but this time we have five vertices. All the vertices in the square can be considered to be the same, so there is only one place that the first edge of the new matching can go up to symmetry. This leaves two places that the second edge can go. First, it could go diagonally across the square, giving RRR_3 (Figure 6.39.3). The other place it could go is parallel to an existing matching edge, giving RRR_4 (Figure 6.39.4).

We now start from a square again, but add in two extra vertices. The two vertices on the square that the last red matching edges are incident with can either be on a diagonal of the existing square (RRR_5 , Figure 6.39.5), or be on an edge of the existing square (RRR_6 , Figure 6.39.6). This finishes the case any two of the matchings form a square.

We now start from a double (Figure 6.15.2). Without increasing the number of vertices, we can either add our third matching in parallel to the existing matchings, giving RRR_7 (Figure 6.39.7), or we could add the third matching so that it is edge-disjoint from the existing matching, giving a square (RRR_1 , Figure 6.39.1).

We now start from a double, but add in one extra vertex. The first edge of the third red matching can only go in one place, namely from the new vertex to an existing vertex. This leaves two possible places for the second edge to go. We can either add it in parallel to the avoided matching edge (RRR_8 , Figure 6.40.1), or so that it is incident with both existing parallel classes (RRR_9 , Figure 6.40.2).

This time, starting from the double, we add in two extra vertices. If the two edges of the third matching go to the same parallel class in the double, the avoided parallel class gives a contradiction to Lemma 6.3.5. So the edges must go to distinct parallel classes, giving RRR_{10} (Figure 6.40.3). This finishes the case of any two of the matchings being a double.

We now start from a join (Figure 6.15.3), and stay on five vertices. To make describing the placement of the third red matching easier, we label the vertices along the path 1 through 5.

First, we place the first edge of the third red matching between 1 and 5. This leaves two places for the second edge to go, either between 3 and 4 (RRR_{11} , Figure 6.40.4), or between 2 and 4 (RRR_{12} , Figure 6.40.5).

Next, we place the first edge of the third red matching between 1 and 4. This leaves three places for the second edge to go. We could place it between 2 and 3, giving a square (RRR_4 , Figure 6.39.4), or we could place it between 2 and 5, giving RRR_{13} (Figure 6.40.6). Finally, we could place it between 3 and 5, giving RRR_{14} (Figure 6.40.7).

Next, we place the first edge of the third red matching between 1 and 3. There are three locations to place the final edge of the third matching, two of which we have already seen. These are between 2 and 4 (RRR_3 , Figure 6.39.3), and between 2 and 5 (RRR_{14} , Figure 6.40.7). Finally, we could place the final edge of the third red matching between 4 and 5, giving RRR_{15} (Figure 6.41.1).

We now place the first edge of the final matching between 1 and 2. This also gives three locations for the final edge, two of which we have already seen. These are between 3 and 4 (RRR_9 , Figure 6.40.2), and between 3 and 5 (RRR_{15} , Figure 6.41.1). Finally, we could place it between 4 and 5, giving RRR_{16} (Figure 6.41.2).

There are all possible places, up to symmetry, for the third red matching on five vertices, as now any placement would have to avoid both 1 and 5, leaving only three vertices.

We now start from a join, and add in one extra vertex. By symmetry, the edge of the third red matching that uses this vertex must go to either 2 or 3 (as otherwise we would have a contradiction to Lemma 6.3.5).

We start by joining it to 2. This leaves six locations for the final edge of the third red matching to go. If we place it either between 1 and 5 or 1 and 3, we get a contradiction to Lemma 6.3.5. The remaining four locations for the final edge of the third red matching are between 1 and 4 (RRR_{17} , Figure 6.41.3), between 3 and 5 (RRR_{18} , Figure 6.41.4), between 4 and 5 (RRR_{19} , Figure 6.41.5), and between 3 and 4 (RRR_{20} , Figure 6.41.6).

So now we place the first edge of the third red matching so that it meets the join at 3. There are now four locations up to symmetry that we could place the final edge of the third matching. If we place it between 1 and 2, we have a contradiction to Lemma 6.3.5. The remaining three locations are between 1 and 5 (RRR_{21} , Figure 6.41.7), between 1 and 4 (RRR_{22} , Figure 6.42.1), and between 2 and 4 (RRR_{23} , Figure 6.42.2).

Now we start from a join, and add in two extra vertices. As already discussed, the third matching edges must meet the join at 2, 3, or 4. If they meet the join at 2 and 3, we have a contradiction to Lemma 6.3.5. By symmetry, they cannot meet the join at 3 and 4. If they meet the join at 2 and 4, we get RRR_{24} (Figure 6.42.3). This finishes the case of any two of the red matchings forming a join.

Finally, we start from a parallel (Figure 6.15.4), firstly having no extra vertices. If we place the first edge of the third matching in the existing parallel class, there are two locations for the second edge to go. First, we could place it parallel to one of the existing edges, giving RRR_8 (Figure 6.40.1). Secondly, we could place it so that it joins the ends of the existing path, giving RRR_{25} (Figure 6.42.4). If we place the first edge so that it is not in the existing parallel class, we must place it so that it joins the parallel class to the path. If we connect to an end of the path, the second edge of the third red matching could go in three places. First, we could place it so that it is incident with the parallel class and the centre of the path of matchings, giving RRR_4 (Figure 6.39.4). Second, we could place it so that it is incident with the parallel class and the available end of the path of matchings, giving RRR_{11} (Figure 6.40.4). Finally, we could place it parallel to an existing matching edge in the path of matchings, giving RRR_{16} (Figure 6.41.2). Finally, if we place the third red matching so that neither edge of the red matching joins the existing parallel component to an end of the path of matchings, we only have RRR_{15} (Figure 6.41.1).

We now start from a parallel and add in one extra vertex. The edge in the third matching that uses this extra vertex must go to either the parallel class or the centre of the path, as otherwise we would have a contradiction to Lemma 6.3.5. We start by joining to the parallel class. The final edge in the third matching can go in four different places. If we place it from the parallel class to an end of the path, we have a contradiction to Lemma 6.3.5. If we

place it from the parallel class to the centre of the path, we get RRR_{20} (Figure 6.41.6). The other two locations for the final edge of the third matching are joining the ends of the path (RRR_{26} , Figure 6.42.5) and parallel to one of the path edges (RRR_{27} , Figure 6.42.6). If we now add the first edge of the third red matching so that it goes to the centre of the path, we have three places for the final edge to go. If we join the ends of the path, we have a contradiction to Lemma 6.3.5. If we join the parallel class to the matching, we get RRR_{19} (Figure 6.41.5). Finally, we could place the final edge in the existing parallel class, giving RRR_{28} (Figure 6.42.7).

For the last case, we start from a parallel and add in two extra vertices. As discussed earlier, each edge of the third matching must either go to the parallel class or the centre of the path. If both edges go to the parallel class, we get a contradiction to Lemma 6.3.5. If one edge goes to the parallel class and the other edge goes to the centre of the path, we get RRR_{29} (Figure 6.42.8). This finishes the entire case check for three red matchings.

We now go through each configuration in turn and check for the existence of $U_{2,4}$ -restrictions containing all three of the extension elements. These checks are done by the `lines()` function from Appendix 6.A.4. The function calls, and an interpretation of the output are given below.

RRR_1 `lines(4, [[0, 5], [0, 5], [1, 4]])` outputs that there is no line containing the three extension elements and a clique element.

RRR_2 This is XS (Figure 6.10.1), and so is forbidden.

RRR_3 This is XG (Figure 6.11.1), and so is forbidden.

RRR_4 `lines(5, [[1, 9], [2, 8], [5, 8]])` outputs that there is no line containing the three extension elements and a clique element.

RRR_5 This contains a TT (Figure 6.9.3), and so is forbidden.

RRR_6 This contains a TT (Figure 6.9.3), and so is forbidden.

RRR_7 This is TM (Figure 6.10.2), and so is forbidden.

RRR_8 This is AF (Figure 6.11.3), and so is forbidden.

RRR_9 This is MW (Figure 6.10.4), and so is forbidden.

RRR_{10} This contains a TT (Figure 6.9.3), and so is forbidden.

RRR_{11} This is DM (Figure 6.10.5), and so is forbidden.

RRR_{12} `lines(5, [[0, 7], [1, 9], [5, 8]])` outputs that there is no line containing the three extension elements and a clique element.

RRR_{13} This is SW (Figure 6.11.5), and so is forbidden.

RRR₁₄ `lines(5, [[1, 9], [2, 6], [5, 8]])` outputs that there is no line containing the three extension elements and a clique element.

RRR₁₅ This is PL (Figure 6.11.4), and so is forbidden.

RRR₁₆ This is BC (Figure 6.10.3), and so is forbidden.

RRR₁₇ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₁₈ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₁₉ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₀ `lines(6, [[0, 14], [5, 14], [7, 13]])` outputs that there is no line containing the three extension elements and a clique element.

RRR₂₁ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₂ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₃ This is DI (Figure 6.11.6), and so is forbidden.

RRR₂₄ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₅ This is AE (Figure 6.11.2), and so is forbidden.

RRR₂₆ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₇ This contains a TT (Figure 6.9.3), and so is forbidden.

RRR₂₈ `lines(6, [[0, 11], [0, 13], [0, 14]])` outputs that there is no line containing the three extension elements and a clique element.

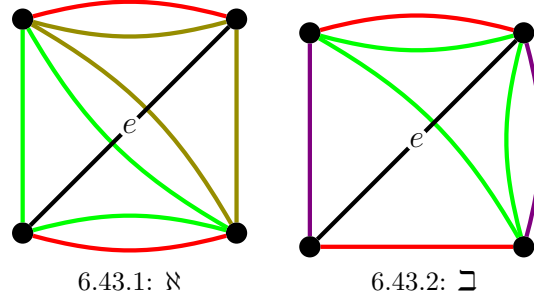
RRR₂₉ This contains a TT (Figure 6.9.3), and so is forbidden.

So there are no configurations of interest.

6.5.3.5 Summary

We have now considered all possible ways to construct a $U_{2,4}$ -restriction where exactly three of the elements are non-clique elements. There are two ways that this can occur, GGR₃ (Figure 6.36.3), and GRR₁ (Figure 6.37.1).

To summarise, there are four configurations in an augmented clique which may quadrate to a $U_{2,4}$ -restriction containing a clique element in the corresponding matroid. These are a green triangle, a doubled-up red matching, and the two configurations (\bowtie and \beth) from Figure 6.43.

Figure 6.43: $U_{2,4}$ -restrictions

6.6 Proof of Theorem 6.1.4

In order to prove Theorem 6.1.4, we need to find an element of the spanning clique to contract. We want this element to only be on three-point lines. We start by restricting the location of \aleph (Figure 6.43.1) and \beth (Figure 6.43.2) configurations.

Lemma 6.6.1. *Let e be in a \aleph (Figure 6.43.1) configuration \aleph_e . It is not possible to have an edge f that is vertex-disjoint from \aleph_e and is in a $U_{2,4}$ -restriction.*

Proof. Assume for a contradiction that this does occur. So we have an edge f such that f is vertex-disjoint from \aleph_e and f is in a $U_{2,4}$ -restriction.

There are four possible configurations that f could be in, in order for f to be in a $U_{2,4}$ -restriction. First, it could either be on a green triangle or on a doubled-up red matching. It could also be in a \aleph (Figure 6.43.1) or a \beth (Figure 6.43.2).

If f was on a green triangle, then one of the red matching edges in \aleph_e would not touch this green triangle, contradicting Lemma 6.3.4.

If f was on a red matching, then this red matching edge would be vertex-disjoint from a green triangle in \aleph_e , contradicting Lemma 6.3.4.

If f was on either a \aleph or a \beth , then we would need to place a red matching so both ends of f and the two vertices in \aleph_e that are not incident with e . However we place this red matching, we are able to find a forbidden TT (Figure 6.9.3) configuration with the red matching in \aleph_e .

As all possibilities have been exhausted, it is impossible for f to be vertex-

disjoint from \aleph_e and in a $U_{2,4}$ -restriction, as required. \square

Lemma 6.6.2. *Let e be in a \sqsupset (Figure 6.43.2) configuration \sqsupset_e . It is not possible to have an edge f that is vertex-disjoint from \sqsupset_e and is in a $U_{2,4}$ -restriction.*

Proof. Assume for a contradiction that this does not occur. So we have an edge f such that f is vertex-disjoint from \sqsupset_e and f is in a $U_{2,4}$ -restriction.

As in the proof of Lemma 6.6.1, there are four possible configurations that f could be in, in order for f to be in a $U_{2,4}$ -restriction. First, it could either be on a green triangle or on a doubled-up red matching. It could also be in a \aleph (Figure 6.43.1) or a \sqsupset (Figure 6.43.2).

If f was on a green triangle, then one of the red matching edges in \sqsupset_e would not touch this green triangle, contradicting Lemma 6.3.4.

If f was on a red matching, then this red matching edge would be vertex-disjoint from the green triangle in \sqsupset_e , contradicting Lemma 6.3.4.

If f was on either a \aleph or a \sqsupset , then we would need to place a red matching so that it touches two vertices of the green triangle in \sqsupset_e and both ends of f . Up to symmetry, there are two ways to do this, both of which contain a forbidden TT (Figure 6.9.3) configuration with a red matching in \sqsupset_e .

As all possibilities have been exhausted, it is impossible for f to be vertex-disjoint from \sqsupset_e and in a $U_{2,4}$ -restriction, as required. \square

Lemma 6.6.3. *Let M be minor-minimal counterexample to Theorem 6.1.4 such that M is maximum sized. There must be an element of a spanning clique of M that is not in a $U_{2,4}$ -restriction.*

Proof. Assume for a contradiction that every clique element is in a $U_{2,4}$ -restriction. Arbitrarily pick e from the clique. From Lemmas 2.3.1 and 6.1.5 we know that the clique must have at least six vertices.

There are four possible configurations that e could be in. First, it could either be on a green triangle or on a doubled-up red matching. It could also be in a \aleph (Figure 6.43.1) or a \sqsupset (Figure 6.43.2).

Assume first that e is on a \aleph or a \sqsupset . As we are on at least six vertices, we are able to find an edge f that is vertex disjoint from the configuration containing e . It follows from Lemmas 6.6.1 and 6.6.2 that this is a contradiction.

Hence e cannot be in a \aleph or a \beth and we can assume that there are no \aleph s or \beth s for the rest of this analysis.

Now we assume that e is on a doubled-up red matching, and consider f as before. As before, f cannot be on a red matching, as this would contradict Lemma 6.3.5. Likewise, if f was in a green triangle, then only one of the two existing parallel classes of red matchings would touch this new green triangle, contradicting Lemma 6.3.4. So e cannot be on a doubled-up red matching.

Now we assume that e is in a green triangle T_e . Because the previous cases have considered everything else, every edge of our complete graph must be in a green triangle. Consider an edge f that is vertex-disjoint from T_e , and consider a green triangle T_f containing f . T_e and T_f must meet at a vertex v , as otherwise we would have a forbidden TL (Figure 6.8.3).

If we have seven or more vertices, then we can find an edge g that is vertex-disjoint from both T_e and T_f . Now we can find a green triangle containing g , and this green triangle must touch both T_e and T_f , and so must contain v . Hence we have a forbidden RA (Figure 6.8.6), and so we have at most six vertices. It follows from Lemma 6.1.5 that we have at least six vertices, and hence we have exactly six vertices.

Let u be the vertex that is not in T_e or T_f . Let T_e have vertices v , v_e , and u_e . Let T_f have vertices v , v_f , and u_f . The edge uv_e must be in a green triangle T_x . Because we cannot have vertex-disjoint green triangles by TL (Figure 6.8.3), up to symmetry, T_x must contain either v or v_f . If T_x contains v_f , then we can find a forbidden TB (Figure 6.8.4), so T_x must consist of the vertices vv_e . Likewise, any green triangle T_y containing the edge uv_f must consist of the vertices vv_f .

We now consider a green triangle T_z containing the edge $u_e u_f$. This green triangle must contain either u or v , as otherwise we would have vertex-disjoint green triangles, which are forbidden by TL (Figure 6.8.3). If T_z contains u , then the green triangles T_f , T_y , and T_z form a forbidden OM (Figure 6.8.2). Hence T_z must consist of the vertices $u_e u_f v$, and the five green triangles T_e , T_f , T_x , T_y , and T_z form a forbidden VQ (Figure 6.8.5).

As we covered every case, it is not possible for every clique element to be in a $U_{2,4}$ -restriction, and so there must be an element of the clique that is not in a $U_{2,4}$ -restriction, as desired. \square

We now have two cases to consider in order to prove Theorem 6.1.4. The first case is when a clique element that is not on a $U_{2,4}$ -restriction is on three non-clique three-point lines.

If a clique element e is not in a $U_{2,4}$ -restriction and is on three non-clique three-point lines, then it must be in one of the four configurations from Figure 6.34. These configurations are the line-star configuration, the two-parallel configuration, the matching-star configuration, and the Betsy Ross configuration.

We now analyse the four configurations.

Definition 6.6.4. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. A **semi-line-star configuration** is a restriction $C' = (G', \mathcal{A}')$ of C , where

- (i) G' is a complete subgraph of G ,
- (ii) G' is spanned by a subgraph consisting of H and $e = uv$, where
 1. H is a star, and
 2. e is vertex-disjoint from H .

In addition, \mathcal{A}' consists of a collection of green triangles, where

- (i) each green triangle is incident with either u or v ,
- (ii) each green triangle contains an edge of H ,
- (iii) every edge of H is in one or two green triangles, and
- (iv) no two members of \mathcal{A}' have the same edge set.

A semi-line-star configuration is **maximal** if it is not a restriction of a semi-line-star configuration (apart from itself).

A **line-star configuration** is a semi-line-star configuration where every edge in H is in two green triangles in \mathcal{A}' .

A schematic is given in Figure 6.44.

Lemma 6.6.5. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-line-star configuration within C , and assume that C_1 has a restriction $C_2 = (G_2, \mathcal{A}_2)$, such that C_2 is a line-star configuration containing at least six vertices. Then there are no green triangles in $\mathcal{A} \setminus \mathcal{A}_1$.

Proof. We use the labels from Figure 6.44, where u, v, s, l_1, l_2 , and l_3 are in G_2 . Let T be a green triangle in $\mathcal{A} \setminus \mathcal{A}_1$. If T does not touch s , then we

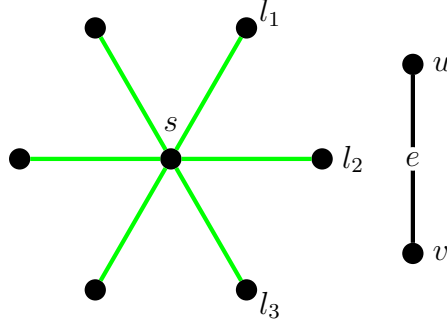


Figure 6.44: Line-star configuration

are able to find a TL (Figure 6.8.3), which is forbidden, so T must touch s . Choose two leaves (l_1 and l_2) in the star, with $T_1 = l_1su$ and $T_2 = l_2sv$ as green triangles. T_1 and T_2 are edge-disjoint, so T must share an edge with one of T_1 or T_2 , as if it does not, we will be able to find a RA (Figure 6.8.6), which is forbidden. Without loss of generality, T shares an edge with T_1 , and so T contains either u or l_1 .

We consider u first. So $T = sux$. If $x = v$, then e is on a green triangle and hence is in a $U_{2,4}$ -restriction, which is forbidden by assumption. If $x = l_i$ for $i \in \{1, 2, 3\}$, then we have a Δ (Figure 6.24.2) configuration, with one of the green triangles doubled up. In this case, `is_forbidden(5, [[0, 1, 4], [0, 3, 6], [1, 2, 7], [2, 3, 9], [2, 3, 9]])` reports that this configuration is forbidden. So x cannot be s , l_i , u , or v . If x is not in G_1 , then we contradict the maximality of C_1 . So x is in G_1 . The maximality of C_1 means that T is identical to, but distinct from, a green triangle that is already in \mathcal{A}' . In this case, we can find a forbidden configuration consisting of a maximal semi-line-star configuration with star S on six vertices with five green triangles, with an edge f of S that is only on one green triangle. The green triangle containing f is doubled-up, giving us the function call `is_forbidden(6, [[1, 3, 10], [1, 3, 10], [5, 7, 10], [5, 8, 11], [9, 10, 12], [9, 11, 13]])`, which reports that this configuration is forbidden.

Hence T cannot touch u , and so $T = sl_1x$. By symmetry, $x \neq u$ and $x \neq v$. If x is not in G_2 , then the three green triangles T , sl_2u , and sl_3v form a RA (Figure 6.8.6), which is forbidden. Hence, without loss of generality, $x = l_2$. So we have the

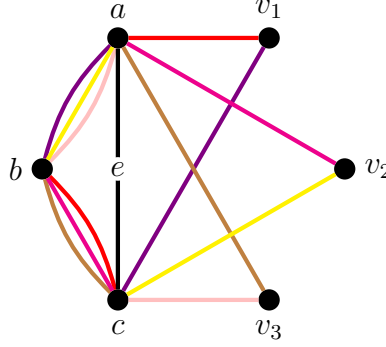


Figure 6.45: Two-parallel configuration

configuration shown in Figure 6.26.2 with an additional green triangle $\{2, 4, 13\}$. The function call `is_forbidden(6, [[0, 1, 5], [0, 2, 6], [0, 4, 8], [1, 3, 10], [2, 3, 12], [2, 4, 13], [3, 4, 14]])` outputs `True`, which means that this configuration is forbidden.

As all possibilities have been exhausted, there can be no green triangles in $\mathcal{A} \setminus \mathcal{A}_1$. \square

Lemma 6.6.6. *Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-line-star configuration within C , and assume that C_1 has a restriction $C_2 = (G_2, \mathcal{A}_2)$, such that C_2 is a line-star configuration containing at least six vertices. Then there are no red matchings in $\mathcal{A} \setminus \mathcal{A}_1$.*

Proof. Let R be a red matching in $\mathcal{A} \setminus \mathcal{A}_1$. Because of Lemma 6.3.4, both edges of R must touch all the green triangles in \mathcal{A}_1 . This means that both edges of R must either touch s or be e .

Assume that one edge of R is e . No matter where the other edge of R is, as it must touch s , we are able to find a forbidden KF (Figure 6.12.2).

Hence both edges of R must touch s , which is impossible. \square

Definition 6.6.7. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. A **semi-two-parallel configuration** is a restriction $C' = (G', \mathcal{A}')$ of C , where

- (i) G' is a complete subgraph of G , and
- (ii) G' is spanned by the vertices $\{a, b, c, v_1, \dots, v_n\}$.

Let e be the edge ac . In addition, \mathcal{A}' consists of a collection of red matchings, where

- (i) each red matching contains either ab or bc ,
- (ii) if a red matching contains ab , then it also contains cv_i , for some $i \in 1, \dots, n$,
- (iii) if a red matching contains bc , then it also contains av_i , for some $i \in 1, \dots, n$,
- (iv) All red matchings in \mathcal{A}' are distinct, and
- (v) each v_i is in either one or two red matchings.

A semi-two-parallel configuration is **maximal** if it is not a restriction of a semi-two-parallel configuration (apart from itself).

A **two-parallel configuration** is a semi-two-parallel configuration where every v_i in G' is contained in two red matchings in \mathcal{A}' .

A schematic is given in Figure 6.45.

Lemma 6.6.8. *Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-two-parallel configuration within C , and assume that C_1 has a restriction $C_2 = (G_2, \mathcal{A}_2)$ such that C_2 is a two-parallel configuration containing at least six vertices. Then there are no green triangles in $\mathcal{A} \setminus \mathcal{A}_1$.*

Proof. We use the labels from Figure 6.45, where a, b, c, v_1, v_2 , and v_3 are in G_2 . Let T be a green triangle in $\mathcal{A} \setminus \mathcal{A}_1$.

First, assume that T consists of three vertices from $G \setminus G_1$. But this is a contradiction to Lemma 6.3.4.

There is no vertex in G_1 that every red matching edge from \mathcal{A}_2 touches, so it is impossible for T to contain two vertices from $G \setminus G_1$ without contradicting Lemma 6.3.4.

The only edge of G_1 that all red matching edges touch is ac . However, this is also e , and hence cannot be on a green triangle as that would cause e to be in a $U_{2,4}$ -restriction, which is forbidden by assumption. Therefore there is no way that T can contain one vertex from $G \setminus G_1$ without contradicting Lemma 6.3.4.

Therefore T must be on vertices from G_1 . Every K_3 subgraph of G_1 that all

red matching edges from \mathcal{A}_1 touch contains e , which is a contradiction to the definition of e .

Hence there are no green triangles in $\mathcal{A} \setminus \mathcal{A}_1$. \square

Lemma 6.6.9. *Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-two-parallel configuration within C , and assume that C_1 has a restriction $C_2 = (G_2, \mathcal{A}_2)$ such that C_2 is a two-parallel configuration containing at least six vertices. Then there are no red matchings in $\mathcal{A} \setminus \mathcal{A}_1$.*

Proof. We use the labels from Figure 6.45, where a, b, c, v_1, v_2 , and v_3 are in G_2 . Let R be a red matching in $\mathcal{A} \setminus \mathcal{A}_1$.

First note that Lemma 6.3.5 implies that R cannot use three or more vertices from $G \setminus G_1$.

Assume that R contains two vertices, u and v , from $G \setminus G_1$. Because of Lemma 6.3.5, the edges of R must touch all red matching edges from the red matchings in \mathcal{A}_1 . Hence, up to symmetry, $R = (ua, vc)$ and we can find a forbidden TT (Figure 6.9.3) using R and red matching from \mathcal{A}_1 .

Now assume that $R = (r_1, r_2)$ contains one vertex v from $G \setminus G_1$. The vertices in G_1 that touch every red matching in \mathcal{A}_1 are a, b , and c . Hence r_2 , the edge of R that uses v , must meet G_1 at one of a, b , or c . We now consider the possible locations for r_1 . If r_1 is only incident with vertices in $\{v_1, \dots, v_n\}$, then it can touch only four different red matchings, which is not enough to satisfy Lemma 6.3.5 as \mathcal{A}_2 consists of at least six red matchings. Hence r_1 must be incident with at least one of a, b , or c . We consider the cases in turn.

First, we consider the case where r_2 meets G_1 at a . Note that this case is the same as when r_2 meets G_1 at c . First, it could be that $r_1 = bc$. This contradicts the maximality of C_1 . Next, consider the case where $r_1 = bv_i$. In this case, we are able to find a forbidden TT (Figure 6.9.3) that uses R and a red matching from \mathcal{A}_1 that avoids v_i and does not contain ab . Finally, we consider the case where $r_2 = cv_i$. In this case, we are able to find a forbidden TT (Figure 6.9.3) consisting of R and a red matching from \mathcal{A}_1 that contains bc and avoids v_i .

Therefore r_2 meets G_1 at b . First, we consider the case that $r_1 = ac$. This

gives us a forbidden XG (Figure 6.11.1) consisting of R and any two red matchings from \mathcal{A}_1 that form a RR-square (Figure 6.17.3). Hence, up to symmetry, $r_1 = av_i$. In this case, we are able to find a forbidden TT (Figure 6.9.3) that consists of R and a red matching from \mathcal{A}_1 of the form (av_j, bc) , where $i \neq j$.

So $R = (r_1, r_2)$ contains only vertices from G_1 . By Lemma 6.3.5, both r_1 and r_2 must touch every red matching in \mathcal{A}_1 . Hence both r_1 and r_2 must touch at least one of a , b , or c . We start by considering the case where r_1 touches two of a , b , and c .

If $r_1 = ac$, then, by Lemma 6.3.5, $r_2 = bv_i$ for some $i \in 1, \dots, n$. In this case, we can find a forbidden XG (Figure 6.11.1). Now assume that $r_1 = ab$. By Lemma 6.3.5, r_2 must touch c . So, by the maximality of C_1 , R must be parallel to an existing red matching. We can easily find a forbidden MW (Figure 6.10.4) in this case. This case is identical to the case where $r_1 = bc$.

Therefore both r_1 and r_2 must touch exactly one vertex from $\{a, b, c\}$. First, we consider the case where $r_1 = ax$ and $r_2 = cy$, where $\{x, y\} \subset \{v_1, \dots, v_n\}$. No matter what x and y are, we are always able to find a forbidden TT (Figure 6.9.3) consisting of R and a red matching from \mathcal{A}_1 that avoids both x and y . Hence, without loss of generality, $r_1 = ax$ and $r_2 = by$, where $\{x, y\} \subset \{v_1, \dots, v_n\}$. In this case, there exists a matching edge cz , where $z \in \{v_1, \dots, v_n\} \setminus \{x, y\}$ that avoids R , a contradiction to Lemma 6.3.5.

Hence there are no red matchings in $\mathcal{A} \setminus \mathcal{A}_1$. \square

Definition 6.6.10. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. A **matching-star configuration** is a restriction $C' = (G', \mathcal{A}')$ of C , where

- (i) G' is a complete subgraph of G ,
- (ii) G' is spanned by a subgraph consisting of H and e , where
 1. H is a star with core s , and
 2. e is an edge that is vertex-disjoint from H .

In addition, \mathcal{A}' consists of a collection of red matchings, where

- (i) each red matching contains e ,
- (ii) each red matching contains an edge of H , and
- (iii) every edge of H is in exactly one red matching.

A matching-star configuration is **maximal** if it is not a restriction of a

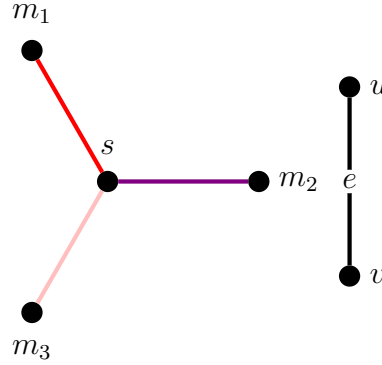


Figure 6.46: Matching-star configuration

matching-star configuration (apart from itself).

A schematic is given in Figure 6.46.

Lemma 6.6.11. *Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C' = (G', \mathcal{A}')$ be a maximal matching-star configuration within C . Let e , s and H be from the definition of G' , and let $X = G \setminus \{s, u, v\}$. Then the end-vertices of e can be labelled u and v in such a way that one of the following holds.*

- (i) (a) $\mathcal{A} \setminus \mathcal{A}'$ contains no red matchings,
- (b) if T is a green triangle from \mathcal{A} , then $T = sux$ for some $x \in X$, and
- (c) no two green triangles in \mathcal{A} have the same edge set.
- (ii) (a) \mathcal{A} contains no green triangles,
- (b) if R is a red matching from $\mathcal{A} \setminus \mathcal{A}'$, then $R = (su, vx)$ for some $x \in X$, and
- (c) no two red matchings in $\mathcal{A} \setminus \mathcal{A}'$ have the same edge set.

Proof. Throughout this proof, we use the labels from Figure 6.46. Note that C' must have at least six vertices.

If $\mathcal{A} = \mathcal{A}'$, then the lemma vacuously holds. Hence we can assume that $\mathcal{A} \neq \mathcal{A}'$.

First assume that \mathcal{A} contains a green triangle T . We need to show that $T = sux$ for some $x \in X$.

Sublemma 6.6.11.1. *Up to swapping the labels on u and v , $T = sux$ for some $x \in X$.*

Subproof. If T does not touch at least one of u or v , then we have a contradiction to Lemma 6.3.4. The same thing occurs if T does not touch s . Note that T cannot touch both u and v , as we are assuming that e is not in a $U_{2,4}$ -restriction. Hence, without loss of generality, $T = suz$ for some $z \in G \setminus \{s, u, v\}$, and we have the desired result. \square

Sublemma 6.6.11.1 is not enough, however. We need every green triangle in \mathcal{A} to be of the form sux .

Sublemma 6.6.11.2. *Assume that there are two distinct green triangles, T and T' in \mathcal{A} . Then up to swapping the labels on u and v , $T = sux$ and $T' = suy$, where $\{x, y\} \subseteq X$.*

Subproof. From Sublemma 6.6.11.1, we can assume that $T = sux$ for some $x \in X$. Also, $T' = swy$, where $w \in \{u, v\}$ and $y \in X$. We first show that $y \neq x$.

Assume that $y = x$, that is, $T' = swx$. If $w = u$, then the two green triangles T and T' , along with the red matching (uv, sz) , for some $z \in X \setminus \{x\}$ form a forbidden CC (Figure 6.12.4). So $T' = svx$. In this case, the two green triangles T and T' , and the red matching (uv, sz) , for some $z \in X \setminus \{x\}$ form a KF (Figure 6.12.2), which is forbidden. Therefore $y \neq x$.

Therefore if the sublemma is false, $T' = svy$. In this case, the two green triangles T and T' , and the red matching (uv, sz) , for some $z \in X \setminus \{x, y\}$, form a forbidden SR (Figure 6.13.5), and hence the sublemma is true. \square

From Sublemmas 6.6.11.1 and 6.6.11.2, we deduce that, if there are no red matchings in $\mathcal{A} \setminus \mathcal{A}'$, then (i) holds.

We set aside the assumption that $\mathcal{A} \setminus \mathcal{A}'$ contains a green triangle for now, and assume that there is one red matching R in $\mathcal{A} \setminus \mathcal{A}'$. We need to show that $R = (su, vx)$ for some $x \in X$.

Sublemma 6.6.11.3. *Up to swapping the labels on u and v , $R = (su, vx)$ for some $x \in X$.*

Subproof. Let $R = (r_1, r_2)$, and let $\{i, j\} = \{1, 2\}$. As every red matching edge must touch every other red matching by Lemma 6.3.5, each edge of R must touch at least one of s , u , or v .

First, assume that R does not touch s . If $r_i = uv$, then r_j cannot touch s , u , or v , which is a contradiction. Hence, without loss of generality, $r_1 = ux$ and $r_2 = vy$ for some $\{x, y\} \subseteq X$. However, in this case, the red matching edge sz , for some $z \in X \setminus \{x, y\}$, does not touch R , which is a contradiction to Lemma 6.3.5.

Therefore R must touch s . Now assume that R does not touch v (note that this is equivalent to R not touching u). Ergo, without loss of generality, $r_1 = ux$ and $r_2 = sy$ for some $\{x, y\} \subseteq X$. If either x or y is not in G' , then we can easily find a forbidden TT (Figure 6.9.3). Hence we can assume that $\{x, y\} \subseteq H \setminus \{s\}$. In this case, the three red matchings R , (uv, sx) , and (uv, sy) form a forbidden BC (Figure 6.10.3).

So R must touch u , v , and s , meaning that the vertex set of R is $\{u, v, s, x\}$ for some $x \in X$. There are three possible red matchings that R could be (uv, sx) , (ux, sv) , or (su, vx) .

First, assume that $R = (uv, sx)$. If $x \notin G'$, then we contradict the maximality of C' , so we can assume that $x \in G$. In this case R and two red matchings from \mathcal{A}' , (uv, sx) and (uv, sy) , for some $y \in H \setminus \{s, x\}$ form a forbidden AF (Figure 6.11.3).

We now assume that $R = (ux, sv)$. We can swap the labels of u and v to get that $R = (su, vx)$.

Hence $R = (su, vx)$, as desired. \square

As before, Sublemma 6.6.11.3 is not enough. We need all red matchings in $\mathcal{A} \setminus \mathcal{A}'$ to be of the form (su, vx) .

Sublemma 6.6.11.4. *Assume that there are two distinct red matchings, R and R' in $\mathcal{A} \setminus \mathcal{A}'$. Then, up to swapping the labels of u and v , $R = (su, vx)$ and $R' = (su, vy)$, where $\{x, y\} \subseteq X$.*

Subproof. From Sublemma 6.6.11.3, we can assume that $R = (su, vx)$ for some $x \in X$. Also, $R' = (st, wy)$, where $\{t, w\} = \{u, v\}$ and $y \in X$. We first show that $y \neq x$.

Assume that $y = x$, that is, $R' = (st, wx)$. If $t = u$, then the three red matchings R , R' , and (uv, sz) , for some $z \in X \setminus \{x\}$ form a MW (Figure 6.10.4), which is forbidden. Therefore $R' = (sv, ux)$. In this case, the three red

matchings R , R' , and (uv, sx) form a forbidden XG (Figure 6.11.1). So $y \neq x$.

Hence, if the sublemma is false, then $R' = (sv, uy)$. In this case, the three red matchings R , R' , and (uv, sz) , for some $z \in X \setminus \{x, y\}$ form a forbidden DI (Figure 6.11.6), and hence the sublemma is true. \square

From Sublemmas 6.6.11.3 and 6.6.11.4, we can surmise that if there are no green triangles in $\mathcal{A} \setminus \mathcal{A}'$, then (ii) holds.

We now have one last thing to show.

Sublemma 6.6.11.5. *There cannot be a red matching R and a green triangle T in $\mathcal{A} \setminus \mathcal{A}'$.*

Subproof. Assume that both R and T are in $\mathcal{A} \setminus \mathcal{A}'$. Without loss of generality, we can assume that $R = (su, vx)$ for some $x \in X$. Then by Sublemma 6.6.11.1, $T = swy$, where $w \in \{u, v\}$ and $y \in X$. We first show that $x \neq y$.

Assume that $x = y$, that is, $T = swx$. If $w = u$, then R , T , and the red matching (uv, sz) , for some $z \in X \setminus \{x\}$ form a forbidden HM (Figure 6.13.4). Hence $T = svx$. In this case, R , T , and the red matching (uv, sz) , for some $z \in X \setminus \{x\}$ form a HN (Figure 6.13.1), which is forbidden. Therefore $x \neq y$.

As C' is maximal, $y \in X$. So $T = swy$ where $\{x, y\} \subseteq X$. If $w = u$, then the red matching edge vx from R does not touch T , a contradiction to Lemma 6.3.4. Ergo $T = svy$. In this case, R , T , and the red matching (uv, sz) for some $z \in X \setminus \{x, y\}$ form a forbidden BP (Figure 6.13.6).

Therefore, if R exist, then T cannot exist, and the result follows. \square

We now have the desired result, as no other possibilities exist. \square

Definition 6.6.12. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. A **Betsy Ross configuration** is a restriction $C' = (G', \mathcal{A}')$ of C , where

- (i) $G' \cong K_4$,
- (ii) G' consists of the vertices a, b, c , and d , and
- (iii) \mathcal{A}' consists of two green triangles and three red matchings, where, up to relabelling,

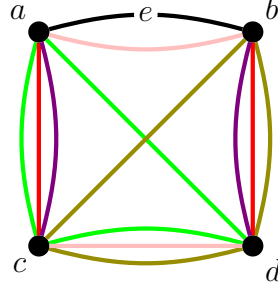


Figure 6.47: Betsy Ross configuration

1. the two green triangles are acd and bcd , and
2. the three red matchings are (ab, cd) , and (ac, bd) twice.

Note that the edge $e = ab$ is in no $U_{2,4}$ -restrictions but is in three non-clique three-point lines.

A representation is given in Figure 6.47.

Lemma 6.6.13. *Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique. Let $C' = (G', \mathcal{A}')$ be a Betsy Ross configuration contained in C . Then $\mathcal{A} = \mathcal{A}'$.*

Proof. We use the labels in Figure 6.47. Let X be an element of $\mathcal{A} \setminus \mathcal{A}'$.

By Lemma 2.3.1, the Betsy Ross configuration is maximal on four vertices as it contains eleven elements, and all matroids that correspond to augmented cliques are simple by definition. Hence X must use at least one vertex from $G \setminus G_1$. X cannot use three or more vertices in $G \setminus G'$, as then we would have a red matching edge in either \mathcal{A}' or X that is vertex-disjoint from a green triangle, contradicting Lemma 6.3.4.

First, X could be a green triangle T . If T uses two vertices from $G \setminus G'$, then T cannot touch all red matching edges in \mathcal{A}' and hence this cannot happen by Lemma 6.3.4.

Hence T can only use one vertex in $G \setminus G'$. In this case, T must contain an edge of G' . As T must touch every red matching edge from the red matchings in \mathcal{A}' , the edge of T from G' must be ad , up to symmetry. In this configuration, we are able to find a forbidden PM (Figure 6.12.3) using T , the green triangle bcd , and one of the (ac, bd) red matchings.

Hence X cannot be a green triangle.

So X must be a red matching $R = (r_1, r_2)$. Consider the case that R uses

two vertices, u and v , from $G \setminus G'$. The edge $f = uv$ cannot be in R , as f does not touch either green triangle in \mathcal{A}' and so is forbidden by Lemma 6.3.4. Hence $R = (ux, vy)$, where $\{x, y\} \subset \{a, b, c, d\}$. This leads to a forbidden TT (Figure 6.9.3), regardless of which vertices x and y are.

Hence there can only be one vertex v in $G \setminus G'$ that R uses. As every edge of R must touch every green triangle in \mathcal{A}' by Lemma 6.3.4, without loss of generality, $r_1 = cv$.

This leaves three possible locations for r_2 . If $r_2 = ab$, then the two green triangles in \mathcal{A}' and R form a forbidden KF (Figure 6.12.2). If $r_2 = ad$, then the three red matchings from \mathcal{A} , (ab, cd) , (ac, bd) , and (ad, cv) form a forbidden XG (Figure 6.11.1). Ergo $r_2 = bd$. But now (ab, cd) and (bd, cv) , and the green triangle acd form a forbidden HM (Figure 6.13.4) in C .

Hence X cannot exist, and so $\mathcal{A} = \mathcal{A}'$, as desired. \square

The name of the Betsy Ross configuration is not a coincidence. If M is a golden-mean matroid with a spanning clique, and M corresponds to an augmented clique that is isomorphic to the Betsy Ross configuration, then M is the Betsy Ross matroid.

To show this, we use the function `is_br()` from Appendix 6.A.6. The function call is `is_br(4, [[0, 2, 4], [3, 4, 5], [0, 5], [0, 5], [1, 4]])`, which returns `True`, as desired.

Lemma 6.6.14. *Let M be a rank- r minimal counterexample to Theorem 6.1.4 such that M is maximum-sized. There is no clique element e of M such that e is on three non-clique three-point lines and no $U_{2,4}$ -restrictions.*

Proof. Let $C = (G, \mathcal{A})$ be a golden-mean augmented clique associated with M . Assume that there is such a clique element e . Then e must be on one of the four configurations considered in Section 6.4.3.

If e is on the Betsy Ross configuration, then Lemma 6.6.13 implies that $|\mathcal{A}| = 5$, and so

$$\binom{r+3}{2} - 5 \leq \epsilon(M) = \binom{r+1}{2} + 5. \quad (6.6.1)$$

As $r \geq 5$, it is impossible for (6.6.1) to be true, so M cannot be a counterexample to Theorem 6.1.4 in this case.

Hence e must be on a line-star, two-parallel, or matching-star configuration.

First, assume that e is in a line-star configuration $C_2 = (G_2, \mathcal{A}_2)$. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-line-star configuration containing C_2 as a restriction. Lemmas 6.6.5 and 6.6.6 tell us that $\mathcal{A} = \mathcal{A}_1$. We now consider the maximum size $\epsilon(M)$ can have. G has three identified vertices, s , u , and v . This leaves $r + 1 - 3 = r - 2$ vertices, each of which determines up to two green triangles. Hence there are at most $2(r - 2)$ elements in \mathcal{A} , for a maximum total of $\binom{r+1}{2} + 2(r - 2)$ elements.

Now assume that e is in a two-parallel configuration $C_2 = (G_2, \mathcal{A}_2)$. Let $C_1 = (G_1, \mathcal{A}_1)$ be a maximal semi-line-star configuration containing C_2 as a restriction. Lemmas 6.6.8 and 6.6.9 tell us that $\mathcal{A} = \mathcal{A}_1$. We now consider the maximum size $\epsilon(M)$ can have. G has three identified vertices, a , b , and c . This leaves $r + 1 - 3 = r - 2$ vertices, each of which determines at most two red matchings. Hence there are no more than $2(r - 2)$ elements in \mathcal{A} , with a maximum of $\binom{r+1}{2} + 2(r - 2)$ elements.

Finally assume that e is in a matching-star configuration $C' = (G', \mathcal{A}')$. Then Lemma 6.6.11 states that there are two possibilities for \mathcal{A}' .

First, it could be that $\mathcal{A} \setminus \mathcal{A}'$ consists of green triangles. In this case, part (i) of Lemma 6.6.11 applies. As before, we consider the maximum size $\epsilon(M)$ could have. In G , there are three distinguished vertices, s , u , and v . This leaves $r + 1 - 3 = r - 2$ vertices, each of which determines at most one red matching in \mathcal{A}' , and at most one green triangle in $\mathcal{A} \setminus \mathcal{A}'$ for a maximum total of $r - 2$ red matchings and $r - 2$ green triangles. Hence the most points that M could have is $\binom{r+1}{2} + 2(r - 2)$.

The other case is when $\mathcal{A} \setminus \mathcal{A}'$ consists of red matchings. In this case, part (ii) of Lemma 6.6.11 applies. As before, there are three distinguished vertices, s , u , and v . Each vertex x in $G \setminus \{s, u, v\}$ determines at most two red matchings, (uv, sx) and (su, vx) , and there are $r - 2$ such vertices. Hence the maximum size of $\epsilon(M)$ is $\binom{r+1}{2} + 2(r - 2)$.

We know that if M is a counterexample, then $\epsilon(M) \geq \binom{r+3}{2} - 5$. So if any of our configurations are to be a counterexample then

$$\binom{r+1}{2} + 2(r - 2) \geq \binom{r+3}{2} - 5 \quad (6.6.2)$$

must be true. However, (6.6.2) implies that $\frac{5}{2}r - 4 \geq \frac{5}{2}r - 2$, which is false. So our configurations cannot be counterexamples.

Therefore no such e can exist. \square

Corollary 6.6.15. *Let M be a rank- r minimal counterexample to Theorem 6.1.4 such that M is maximum-sized. Then every clique element that is not in a $U_{2,4}$ -restriction must be on exactly two non-clique three-point lines.*

Proof. Let e be a clique element of M that is not in any $U_{2,4}$ -restrictions. Since M is a minimal counterexample, we have that $\epsilon(M) \geq \binom{r+3}{2} - 5$, and that $\epsilon(M/e) \leq \binom{r+2}{3} - 5$. Hence

$$\epsilon(M) - \epsilon(M/e) \geq r + 2. \quad (6.6.3)$$

As e is not in any $U_{2,4}$ -restrictions, it follows from (6.6.3) that e is on at least $r + 1$ three-point lines. Therefore e must be on at least two non-clique three-point lines. However, Lemma 6.6.14 states that e can be on no more than two non-clique three-point lines. Hence e must be on exactly two non-clique three-point lines. \square

Corollary 6.6.16. *Let M be a rank- r minimal counterexample to Theorem 6.1.4 such that M is maximum-sized, and let e be a clique element of M that is on no $U_{2,4}$ -restrictions. Then $\text{si}(M/e)$ is isomorphic to a member of \mathcal{G}_{r-1} .*

We now construct all augmented clique representations for all members of \mathcal{G}_r .

Lemma 6.6.17. *Let M be isomorphic to T_r^2 for some $r \geq 3$. Let (G, \mathcal{A}) be a golden-mean augmented clique associated with M . Then \mathcal{A} consists of $2(r-1)$ green triangles. There is an edge e of G such that every green triangle in \mathcal{A} contains e . For every vertex v of G not incident with e , there are exactly two green triangles in \mathcal{A} that contain e and v .*

Proof. Let H be the weighted directed graph that corresponds to a \mathbb{G} -graphic representation of M , as described on Page 32. The loop in H that is incident with all non-trivial parallel classes is the acme of M . Let a be this loop, and

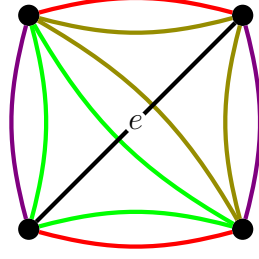


Figure 6.48: Five-point line with four non-clique elements

assume that a is not in G . There must be an element in G that is contained in a leg of M , for otherwise G cannot be spanning. Let b be such an element. Any permutation of the elements of a leg that fixes the acme, when applied to all the legs of M simultaneously, is an automorphism of M . Therefore we can assume that b is a loop in H . Now b is contained in $r - 1$ triangles of G . Since b is contained in $r - 1$ long lines in M , it follows that $\text{cl}_M(\{a, b\})$ contains three elements of G . Let $\{b, c, d\}$ be those three elements, so that in H , the edges that c and d correspond to are incident with both a and b . One of the other triangles of G that contains b also contains f , another loop of H , as well as the edge of H that is incident with b and d . Since $\{b, c, d\}$ is a triangle of G , and $\{b, f\}$ is contained in a triangle of G , it follows that f is contained in a triangle of G with either c or d . But consideration of H shows that no such line exists.

Now we know that G contains a , the acme of M . By again applying an automorphism of M , we can assume that G contains b , which is another loop in H . Since b is contained in $r - 1$ lines in G , an inspection of H shows that all loops of H must be in G , as well as precisely one edge between each pair of vertices. Now the elements of M not in G correspond to two edges from each non-trivial parallel class of H . We see that each triangle of G that contains a spans two elements not in G , and the result follows easily. \square

Lemma 6.6.18. *Let M be a simple golden-mean matroid of rank at least four with a spanning clique, and assume that (G, \mathcal{A}) is a corresponding golden-mean augmented clique. Let ℓ be a five-point line of M . Then there is a K_5 subgraph of G that contains all members of \mathcal{A} that correspond to elements of ℓ .*

Proof. It is clear that ℓ contains no more than three edges from G . If ℓ contains exactly three edges from G , then by the definition of green elements, both non-clique elements of ℓ must be green triangles, and furthermore, they must have the same vertex set, so there is nothing left to prove. Likewise, if ℓ contains exactly two elements from G , then the three non-clique elements of ℓ must be three red matchings sharing the same vertex set, and we are done.

Now assume that ℓ contains four non-clique elements, X_1, \dots, X_4 , as well as e , a clique element. Then the four elements e, X_x, X_y , and X_z must form a $U_{2,4}$ -restriction for all $\{x, y, z\} \subset \{1, 2, 3, 4\}$. All such $U_{2,4}$ -restrictions were discovered in Section 6.5.3.5. From this, we can see that the only way to form a five-point line with four non-clique elements is the configuration shown in Figure 6.48, and again we are done.

Hence we assume that ℓ consists entirely of non-clique elements. Let the elements of ℓ be X_1, \dots, X_5 . Note that any long line containing two or more elements of ℓ must only contain elements of ℓ . Hence, in particular, any two elements of ℓ must not be on a long line with a clique element.

Assume that X_1 and X_2 are green triangles. Then, using the computer checks from Section 6.4.1.2 and the observation above, we can see that X_1 and X_2 must join at a vertex v . Furthermore, any two green triangles that are in ℓ must meet at a vertex. Now consider X_3 . If X_3 is a green triangle, then it must meet both X_1 and X_2 at a single vertex. Hence X_1, X_2 , and X_3 must form either a RA (Figure 6.8.6) or a TB (Figure 6.8.4), both of which are forbidden. So X_3 must be a red matching.

By Lemma 6.3.4, both edges of X_3 must touch both X_1 and X_2 . This means that the edges of X_3 must either be incident with v , or be of the form x_1x_2 , where $x_i \in X_i \setminus \{v\}$. Assume that one edge of X_3 is incident with v . Then X_1, X_2 , and X_3 form either a SR (Figure 6.13.5) or a PM (Figure 6.12.3), both of which are forbidden. Hence X_3 , and, by an identical argument, X_4 and X_5 , must be contained in the K_5 spanned by X_1 and X_2 , as desired.

Therefore we assume that ℓ contains at most one green element. So we can assume that X_1, \dots, X_4 are all red matchings. By considering the computer checks from Section 6.4.1.3, we can assume that X_i and X_j form an alternating path, for all $\{i, j\} \subset \{1, 2, 3, 4\}$. Assume that X_5 is a green triangle.

Then, by the computer checks in Section 6.4.1.4, X_5 must form the configuration shown in Figure 6.16.2 with each of X_1, \dots, X_4 . Hence X_1, X_2 , and X_5 must form either a FS (Figure 6.13.2) or a BP (Figure 6.13.6), both of which are forbidden. Therefore X_5 must be a red matching.

Now consider the three red matchings X_1, X_2 , and X_3 , and assume that X_3 is not contained within the K_5 spanned by $X_1 \cup X_2$. As X_3 must form an alternating path with both X_1 and X_2 , it must be that X_1, X_2 , and X_3 form a forbidden DI (Figure 6.11.6). Hence X_3 must be contained within the K_5 spanned by $X_1 \cup X_2$, and the result follows. \square

Lemma 6.6.19. *Let M be isomorphic to either G_r or HP_r for $r \geq 4$, and let $C = (G, \mathcal{A})$ be a corresponding golden-mean augmented clique. Then C is one of the following three configurations.*

- (i) *C consists of a line-star configuration (see Definition 6.6.4) that spans every vertex of G with the addition of two green triangles, both on the vertices su . In this case, $M \cong G_r$.*
- (ii) *C consists of a two-parallel configuration (see Definition 6.6.7) that spans every vertex of G with the addition of two green triangles, both on the vertices abc . In this case, $M \cong HP_r$.*
- (iii) *C consists of a matching-star configuration (see Definition 6.6.10) that spans every vertex of G with the addition of all possible green triangles as described in part (i) of Lemma 6.6.11, as well as two additional green triangles, both on the vertices su . In this case, $M \cong G_r$.*

Proof. We first show that the lemma is true for low rank.

Sublemma 6.6.19.1. *The lemma is true when $4 \leq r \leq 5$.*

Subproof. We prove this by case analysis. The code in Appendix 6.A.7 creates eighteen augmented cliques at both rank four and rank five. We then create each augmented clique by hand, and the sublemma is verified. \square

Now assume that the lemma is false, and that r is the smallest value for which the lemma fails. Note that, by Sublemma 6.6.19, $r \geq 6$, and hence G has at least seven vertices. Let ℓ be the unique five-point line of M , and recall that G is an arbitrary spanning clique of M . We start by showing that ℓ contains exactly three edges of G .

Sublemma 6.6.19.2. *The unique five-point line ℓ of M contains exactly two non-clique elements.*

Subproof. Assume that this sublemma is false. It is clear that ℓ contains at least two non-clique elements. If ℓ contains exactly three non-clique elements, then these three non-clique elements are spanned by two clique elements, and hence are distinct red matchings with identical edge sets. However, this is a forbidden TM (Figure 6.10.2).

Hence ℓ contains at least four non-clique elements. By Lemma 6.6.18, there is a K_5 -subgraph K of G that contains all members of \mathcal{A} corresponding to elements of ℓ . Let e be an edge of $G \setminus K$ such that e is not incident with any edge of G that is in ℓ . Note that e exists, as at most one element of ℓ is an edge of G , and this edge must be spanned by K , and G has at least seven vertices.

By our choice of e , there is no triangle of M consisting of e , an edge of G that is not in ℓ , and an edge of G that is in ℓ . Assume that T is a triangle of M containing e , an edge e' of G not in ℓ , and an l that is in ℓ . Then l corresponds to a member of \mathcal{A} . Then l is spanned by the edges e and e' , so the member of \mathcal{A} corresponding to l contains the edge e , which is contradictory, since this member of \mathcal{A} should be contained in K . Hence, in M/e , no parallel class contains an element of ℓ and an edge of G .

Now we can assume that ℓ is a five-point line in $\text{si}(M/e)$, and we can assume that $\text{si}(G/e)$ is a spanning clique of $\text{si}(M/e)$ that does not contain three points in L . However, since e is not in five-point line of M , Corollary 2.2.2 tells us that $\text{si}(M/e)$ is isomorphic to G_{r-1} or HP_{r-1} .

Now we can apply induction, which leads to a contradiction, since in the augmented cliques described in the statement of the lemma, the five-point line contains exactly two non-clique elements. \square

Hence G contains exactly three elements of ℓ . Let T be the triangle of G formed by these three elements. Since $|\text{cl}_M(T)| = 5$, we see that \mathcal{A} contains two copies of T . Let these copies be T_1 and T_2 , and let t_1 and t_2 be the corresponding elements of ℓ . Consider the golden-mean augmented clique $C' = (G, \mathcal{A}')$, where $\mathcal{A}' = \mathcal{A} \setminus \{T_1, T_2\}$. Note that C' corresponds to the golden-mean matroid $M' = M \setminus t_1 \setminus t_2$. By inspecting the matrices of G_r

(Figure 2.1.2) and HP_r (Figure 2.1.3), we note that there exists an edge x in T , such that x is in no $U_{2,4}$ -restrictions in M' , and x is contained in $2(r-2)$ three-point lines in M' . As $r \geq 6$ (Sublemma 6.6.19.1 covers the lower ranks), this means that x is on at least three non-clique three-point lines, and we can apply the case analysis from Section 6.4.3, to get that C' is one of the configurations given in Figure 6.34 (with the exception of the Betsy Ross, as $r \geq 6$).

We now consider C , which is the augmented clique C' with the addition of two green triangles, T_1 and T_2 . Note that T_1 and T_2 have the same vertex set T .

First, we assume that C' is a line-star configuration (Definition 6.6.4). Using the labels from Figure 6.44, if $T = \{s, u, v\}$, then we are done. Hence, without loss of generality, $T = \{l_1, u, v\}$. In this case, the three green triangles T_1 , sul_1 , and svl_2 form a forbidden OM (Figure 6.8.2). Therefore $T = \{s, u, v\}$ and case (i) holds. The additional remark that $M \cong G_r$ follows from the fact that there are green elements in M , which equate to $U_{2,4}$ -restrictions.

Now we assume that C' is a two-parallel configuration (Definition 6.6.7). Using the labels from Figure 6.45, if $T = \{a, b, c\}$, then we are done. So, without loss of generality, we can assume that $T = \{a, c, v_1\}$. In this case, the two green triangles T_1 and T_2 , along with the red matching (ab, cv_2) , form a CC (Figure 6.12.4), which is forbidden. So $T = \{a, b, c\}$ and case (ii) holds. The additional remark that $M \cong HP_r$ follows from the fact that there are no green elements in M with the exception of t_1 and t_2 .

Finally we assume that C' is a matching-star configuration (Definition 6.6.10). We can apply either case (i) or case (ii) of Lemma 6.6.11 to C' . If we apply case (ii), we can see that C' is contained within a two-parallel configuration, which was considered above. Hence we must apply case (i). Using the labels from Figure 6.46, if $T = \{s, u, v\}$, then we are done. Thus, without loss of generality, $T = \{u, v, m_1\}$. Now the green triangle T_1 does not touch the red matching edge sm_2 , which is a contradiction to Lemma 6.3.4. Hence $T = \{s, u, v\}$, and case (iii) holds. The additional remark that $M \cong G_r$ follows from the fact that there are at least three green elements in M , which equate to $U_{2,4}$ -restrictions. \square

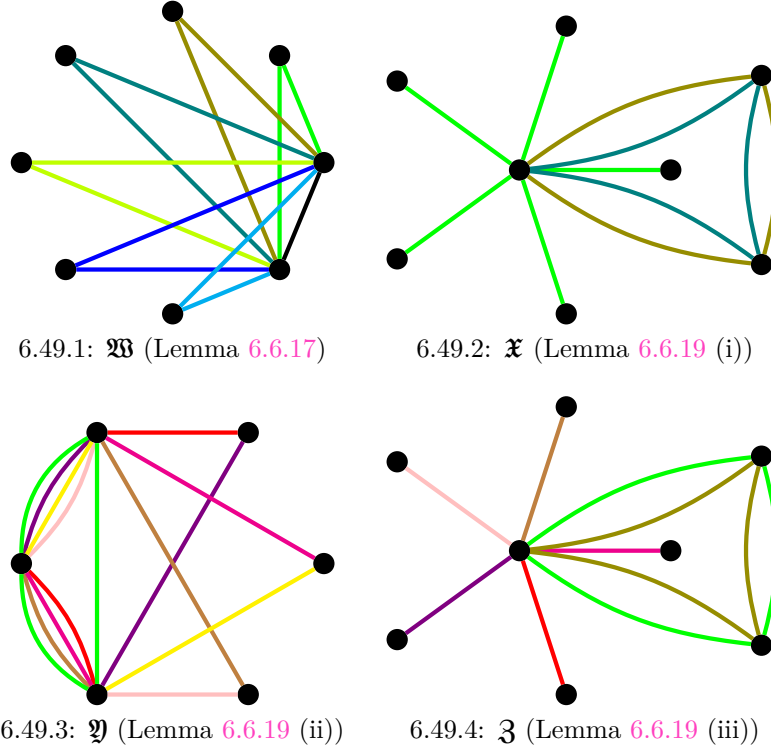


Figure 6.49: Schematics of the four possible outcomes for C_e and C_f

We can now prove Theorem 6.1.4. Let M be a rank- r counterexample to Theorem 6.1.4 with an augmented clique representation $C = (G, \mathcal{A})$ such that M is maximum-sized. Let e be a clique element of M that is in no $U_{2,4}$ -restrictions. Then Corollary 6.6.16 tells us that $M_e = \text{si}(M/e)$ is isomorphic to a member of \mathcal{G}_{r-1} .

Let $C_e = (G_e, \mathcal{A}_e)$ be an augmented clique corresponding to M_e . Now, Lemmas 6.6.17 and 6.6.19 tell us that C_e is one of the four configurations shown in Figure 6.49.

Let f be an edge of G_e that is not in any $U_{2,4}$ -restrictions in M_e . If f were in a $U_{2,4}$ -restriction in M , this restriction would have to contain e , which is a contradiction. Hence f is not in any $U_{2,4}$ -restrictions in M , and we can apply Corollary 6.6.16 to find that $M_f = \text{si}(M/f)$ is also isomorphic to a member of \mathcal{G}_{r-1} .

Let $C_f = (G_f, \mathcal{A}_f)$ be an augmented clique corresponding to M_f . Now, Lemmas 6.6.17 and 6.6.19 tell us that C_f is one of the four configurations shown in Figure 6.49.

First, we assume that C_e is a \mathfrak{B} (Figure 6.49.1). Then we can select f so that it does not touch the vertex identified with e in C_e and is not in any green triangles. Now, upon considering the possible members of \mathcal{A}_f and Lemma 6.3.2, we note that C_f must also be a \mathfrak{B} , and hence C is a \mathfrak{B} . We now construct a $GF(4)$ matrix for M . Let D_n^x be the $n \times \binom{n}{2}$ matrix whose entries consist of all n -tuples with two non-zero entries, with the first being 1 and the second being x . When we consider the star of G where the core is a vertex that is touched by all green triangles of \mathcal{A} , it is not too hard to see that M must be represented over $GF(4)$ by a matrix of the following form.

$$[I_r \mid D_r^1 \mid D_r^\alpha \mid D_r^{\alpha^2}]$$

This matrix is scaling-equivalent to the $GF(4)$ matrix for T_r^2 given in Figure 2.5.1, and hence M is isomorphic to T_r^2 .

We now assume that C_e is a \mathfrak{X} (Figure 6.49.2). Then we can select f so that it does not touch the vertex identified with e in C_e and is not in any green triangles. Now, upon considering the possible members of \mathcal{A}_f and Lemma 6.3.2, we note that C_f must also be a \mathfrak{X} , and hence C is a \mathfrak{X} . We now construct a $GF(4)$ matrix for M . When we consider the star of G where the core is the vertex of G that is the core of the star from the underlying line-star configuration, it is not too hard to see that M must be represented over $GF(4)$ by a matrix of the following form.

$$\mathfrak{M} = \left[\begin{array}{c|c|c|c|c} & & \begin{array}{cc} 1 & 1 \\ \alpha & \alpha^2 \end{array} & \begin{array}{ccc} 0 & \cdots & 0 \\ * & \cdots & * \end{array} & \begin{array}{ccc} * & \cdots & * \\ 0 & \cdots & 0 \end{array} \\ I_r & D_r^1 & \hline 0_2^{r-2} & I_{r-2} & I_{r-2} \end{array} \right]$$

where each $*$ is either α or α^2 . Now, by selecting the correct rows and columns, we can find a rank-three submatrix of \mathfrak{M} for each $*$ in the fourth block of \mathfrak{M} . This submatrix must be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & *_2 \\ 0 & 1 & 0 & 1 & 0 & 1 & \alpha & \alpha^2 & *_1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix represents a rank-three golden-mean matroid with ten elements, and hence, by Lemma 2.3.1, must be isomorphic to either T_3^2 , G_3 , HP_3 , Y_{10} , S_{10} or $BR \setminus p$. As there is a five-point line, we can further restrict this list of possible matroids to T_3^2 , G_3 , or HP_3 . As there are four-point lines, this submatrix must therefore represent G_3 over $GF(4)$. Now, by uniqueness of representation, we can conclude that $*_1 = *_2 = \alpha$.

By repeating this argument, we can discover that

$$\mathfrak{M} = \left[\begin{array}{c|c|cc|ccc} & & 1 & 1 & 0 & \cdots & 0 & \alpha & \cdots & \alpha \\ & & \alpha & \alpha^2 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\ \hline & I_r & D_r^1 & & & & & & & \\ & & & 0_2^{r-2} & & I_{r-2} & & & I_{r-2} & \end{array} \right]$$

which is the $GF(4)$ matrix for G_r given in Figure 2.5.2, and hence $M \cong G_r$.

Next, we consider the case where C_e is a \mathfrak{Y} (Figure 6.49.3). Then we can select f so that it does not touch the vertex identified with e in C_e and is not in any green triangles or red matchings. Now, upon considering the possible members of \mathcal{A}_f and Lemma 6.3.2, we note that C_f must also be a \mathfrak{Y} , and hence C is a \mathfrak{Y} . We now construct a $GF(4)$ matrix for M . When we consider the star of G where the core is the vertex of G that is incident with both sets of parallel classes of red matchings, it is not too hard to see that M must be represented over $GF(4)$ by a matrix of the following form.

$$\mathfrak{M} = \left[\begin{array}{c|c|cc|ccc} & & 1 & 1 & * & \cdots & * & * & \cdots & * \\ & & \alpha & \alpha^2 & * & \cdots & * & * & \cdots & * \\ \hline & I_r & D_r^1 & & & & & & & \\ & & & 0_2^{r-2} & & I_{r-2} & & & I_{r-2} & \end{array} \right]$$

where each $*$ is a non-zero element of $GF(4)$. Now, by selecting the correct rows and columns, we can find a rank-three submatrix of \mathfrak{M} for each column

from the fourth block of \mathfrak{M} . This submatrix must be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & *_1 & *_3 \\ 0 & 1 & 0 & 1 & 0 & 1 & \alpha & \alpha^2 & *_2 & *_4 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix represents a rank-three golden-mean matroid with ten elements, and hence, by Lemma 2.3.1, must be isomorphic to either T_3^2 , G_3 , HP_3 , Y_{10} , S_{10} or $BR \setminus p$. As there is a five-point line, we can further restrict this list of possible matroids to T_3^2 , G_3 , or HP_3 . As there are no further very long lines, this submatrix must therefore represent HP_3 over $GF(4)$. Now, by uniqueness of representation, we can conclude that $*_1 = *_4 = \alpha$ and $*_2 = *_3 = 1$.

By repeating this argument, we can discover that

$$\mathfrak{M} = \left[\begin{array}{c|c|c|c|c} & & \begin{array}{cc} 1 & 1 \\ \alpha & \alpha^2 \end{array} & \begin{array}{ccc} \alpha & \cdots & \alpha \\ 1 & \cdots & 1 \end{array} & \begin{array}{ccc} 1 & \cdots & 1 \\ \alpha & \cdots & \alpha \end{array} \\ I_r & D_r^1 & \hline 0_2^{r-2} & I_{r-2} & I_{r-2} \end{array} \right]$$

which is scaling-equivalent to the $GF(4)$ matrix for HP_r given in Figure 2.5.3, and hence $M \cong HP_r$.

Finally, we assume that C_e is a \mathfrak{Z} (Figure 6.49.4). Then we can select f so that it does not touch the vertex identified with e in C_e and is not in any green triangles or red matchings. Now, upon considering the possible members of \mathcal{A}_f and Lemma 6.3.2, we note that C_f must also be a \mathfrak{Z} , and hence C is a \mathfrak{Z} . We now construct a $GF(4)$ matrix for M . When we consider the star of G where the core is the vertex of G that is the core of the star from the underlying matching-star configuration, it is not too hard to see

that M must be represented over $GF(4)$ by a matrix of the following form.

$$\mathfrak{M} = \left[\begin{array}{c|c|cc|ccc|ccc} & & 1 & 1 & 0 & \cdots & 0 & \star & \cdots & \star \\ & & \alpha & \alpha^2 & * & \cdots & * & \star & \cdots & \star \\ & D_r^1 & \hline & & 0_2^{r-2} & & I_{r-2} & & I_{r-2} \end{array} \right]$$

where each $*$ is either α or α^2 , and each \star is a non-zero element of $GF(4)$ with the added condition that two \star s in the same column have the same value. Now, by selecting the correct rows and columns, we can find a rank-three submatrix of \mathfrak{M} for each $*$ in the fourth block of \mathfrak{M} . This submatrix must be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \star \\ 0 & 1 & 0 & 1 & 0 & 1 & \alpha & \alpha^2 & * & \star \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix represents a rank-three golden-mean matroid with ten elements, and hence, by Lemma 2.3.1, must be isomorphic to either T_3^2 , G_3 , HP_3 , Y_{10} , S_{10} or $BR \setminus p$. As there is a five-point line, we can further restrict this list of possible matroids to T_3^2 , G_3 , or HP_3 . As there are also four-point lines, this submatrix must therefore represent G_3 over $GF(4)$. Now, by uniqueness of representation, we can conclude that $* = \star = \alpha^2$.

By repeating this argument, we can discover that

$$\mathfrak{M} = \left[\begin{array}{c|c|cc|ccc|ccc} & & 1 & 1 & 0 & \cdots & 0 & \alpha^2 & \cdots & \alpha^2 \\ & & \alpha & \alpha^2 & \alpha^2 & \cdots & \alpha^2 & \alpha^2 & \cdots & \alpha^2 \\ & D_r^1 & \hline & & 0_2^{r-2} & & I_{r-2} & & I_{r-2} \end{array} \right].$$

Let \mathfrak{N} be the $GF(4)$ matrix of G_r from Figure 2.5.2. Add the first row of \mathfrak{N} to the second row, and add every other row of \mathfrak{N} scaled by α to the second row of \mathfrak{N} . Now \mathfrak{N} is scaling-equivalent to \mathfrak{M} , and thus $M \cong G_r$.

6.7 Future Work

It is anticipated that our Theorem 6.1.4, when combined with the following theorem due to Geelen and Nelson (personal correspondence) will lead to a proof of Conjecture 1.2.3 for matroids of sufficiently high rank. In this section, we outline how we anticipate this combination will occur.

Theorem 6.7.1. *Let \mathcal{M} be a quadratically dense minor-closed class of matroids and let $p(x)$ be a real quadratic polynomial with positive leading coefficient. If $h_{\mathcal{M}}(n) > p(n)$ for infinitely many $n \in \mathbb{Z}^+$, then for all $r, s \in \mathbb{Z}^+$ there exists $M \in \mathcal{M}$ satisfying $\epsilon(M) > p(r(M))$ and $r(M) \geq r$ such that either*

- (1) *M has a spanning clique restriction, or*
- (2) *M is vertically s -connected and there is an s -element independent set S of M so that $\epsilon(M) - \epsilon(M/e) > p(r(M)) - p(r(M) - 1)$ for each $e \in S$.*

If Conjecture 1.2.3 is false, then the conditions of Theorem 6.7.1 are met. Theorem 6.1.4 shows that outcome (1) cannot occur, and hence outcome (2) must occur. To show that outcome (2) is also impossible, we need to apply the following lemmas, both due to Geelen and Nelson (personal correspondence).

Lemma 6.7.2. *Let $k \in \mathbb{Z}^+$, let M be a matroid and let N be a minor of M such that $\mathcal{T}_k(N)$ is a tangle. If $X \subseteq E(M)$ is contained in a $\mathcal{T}_k(M, N)$ -small set, then there is a minor M' of M such that $M'|X = M|X$, M' has N as a minor, and X is contained in a $\mathcal{T}_k(M', N)$ -small set X' such that $E(M') = E(N) \cup X'$ and $\lambda_{M'}(X') = r_{\mathcal{T}_k(M', N)}(X) = r_{\mathcal{T}_k(M, N)}(X)$.*

Lemma 6.7.3. *There is a function $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ so that, for all $m, n, l, t \in \mathbb{Z}$ with $m > t \geq 0$, $l \geq 2$, and $n \geq f(m, l)$, if $M \in \mathcal{U}(l)$ has an $M(K_{n+1})$ -minor N with corresponding tangle $\mathcal{T} = \mathcal{T}_{\lceil 2n/3 \rceil}(M, N)$ and $X \subseteq E(M)$ satisfies $r_{\mathcal{T}}(X) = t$, then M has a minor M' with an $M(K_{m+1})$ -restriction R so that $X \cap E(R) = \emptyset$, $M'|X = M|X$, $E(M') = E(R) \cup X$ and $\lambda_{M'}(X) = t$.*

Using these two lemmas, we are able to glue the s -element independent set from outcome (2) of Theorem 6.7.1 onto a spanning clique, where we anticipate that our techniques developed in the proof of Theorem 6.1.4 will be able to show that this case is also contradictory.

Appendix 6.A Code

6.A.1 Lemma 6.1.5

```

MS = [Matroid(Matrix(GF(19),matroids.CompleteGraphic(5).
    representation().rows()))]

count = 0
while len(MS) > 0:
    last = []
    MS_hold = []
    for N in MS:
        for M in N.linear_extensions(simple=True, fundamentals=
            funds):
            MS_hold.append(M)
    last = MS
    MS = MS_hold
    print [count, len(MS)]
    count = count + 1

last = get_nonisomorphic_matroids(last)

```

This code starts from $M(K_5)$ and constructs all golden-mean matroids (over $GF(19)$) that have $M(K_5)$ as a restriction. Once there are no more possible extensions, it creates a list, `last` of matroids that are maximum-sized.

Upon running this code (which takes a long time), we discover that Lemma 6.1.5 is indeed true. That is, Theorem 6.1.4 is true when $r = 4$.

6.A.2 Lemma 6.2.1

```

for r in [4, 5, 6]:
    mat = matrix(GF(19), matroids.CompleteGraphic(r).
        representation().rows())
    M = Matroid(mat)
    for N in M.linear_extensions(element='e', fundamentals=funds,
        simple=True):
        CCs = N.circuit_closures()[2]

```

```

if not any(['e' in x for x in CCs]):
    print N

```

This code is used to prove Lemma 6.2.1. We construct all golden-mean single element extensions of $M(K_r)$ for r in [4, 5, 6]. If there were any counterexamples to the lemma, then e would not be in a long line of the constructed matroid. This code prints out any matroids that give counterexamples, and nothing is output, so no counterexamples exist.

6.A.3 Testing For Forbidden Configurations

```

names = ['a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i'] # 9
    extensions should be plenty
def is_forbidden(n, exts): # n is a positive integer, exts is a
    list of lists.
    grab = [Matroid(Matrix(GF(19),matroids.CompleteGraphic(n).
        representation().rows()))]
    hold = []
    for k in range(len(exts)):
        for M in grab:
            for N in M.linear_extensions(simple=True, element=names[k]
                ], fundamentals=funds, F=exts[k]):
                hold.append(N)
    grab = hold
    hold = []
    if len(grab) == 0:
        return True
    else:
        return False

```

This code returns `True` if it is not possible to construct a golden-mean matroid by starting with $M(K_n)$ and extending into all the flats in `exts`, and `False` otherwise.

6.A.4 Finding Lines

```

names = ['a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i'] # 9
    extensions should be plenty

```

```

def lines(n, exts):
    grab = [Matroid(Matrix(GF(19),matroids.CompleteGraphic(n).
        representation().rows()))]
    hold = []
    l = len(exts)
    lines = []
    for k in range(l):
        for M in grab:
            for N in M.linear_extensions(simple=True, element=names[k]
                ], fundamentals=funds, F=exts[k]):
                hold.append(N)
        grab = hold
        hold = []
    if l == 1:
        for 0 in grab:
            lines.append([C for C in 0.circuit_closures()[2] if 'a'
                in C])
    elif l == 2:
        for 0 in grab:
            lines.append([C for C in 0.circuit_closures()[2] if 'a'
                in C and 'b' in C])
    elif l == 3:
        for 0 in grab:
            lines.append([C for C in 0.circuit_closures()[2] if 'a'
                in C and 'b' in C and 'c' in C])
    elif l == 4:
        for 0 in grab:
            lines.append([C for C in 0.circuit_closures()[2] if 'a'
                in C and 'b' in C and 'c' in C and 'd' in C])
    return lines

```

This code takes an augmented clique and constructs all possible golden-mean matroids that arise from that augmented clique. For each of the matroids constructed, `lines` returns the list of all lines that contain all of the extension

elements.

6.A.5 Finding Multiple Lines

```

names = ['a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i'] # 9
    extensions should be plenty
def all_lines(n, exts, e):
    grab = [Matroid(Matrix(GF(19),matroids.CompleteGraphic(n).
        representation().rows()))]
    hold = []
    l = len(exts)
    lines = []
    for k in range(l):
        for M in grab:
            for N in M.linear_extensions(simple=True, element=names[k],
                fundamentals=funds, F=exts[k]):
                hold.append(N)
        grab = hold
        hold = []
    if l == 2:
        for O in grab:
            lines.append([C for C in O.circuit_closures()[2] if e in
                C and ('a' in C or 'b' in C)])
    elif l == 3:
        for O in grab:
            lines.append([C for C in O.circuit_closures()[2] if e in
                C and ('a' in C or 'b' in C or 'c' in C)])
    elif l == 4:
        for O in grab:
            lines.append([C for C in O.circuit_closures()[2] if e in
                C and ('a' in C or 'b' in C or 'c' in C or 'd' in C)])
    elif l == 5:
        for O in grab:
            lines.append([C for C in O.circuit_closures()[2] if e in
                C and ('a' in C or 'b' in C or 'c' in C or 'd' in C or
                    'e' in C)])

```

```

elif l == 6:
    for 0 in grab:
        lines.append([C for C in 0.circuit_closures()[2] if e in
            C and ('a' in C or 'b' in C or 'c' in C or 'd' in C or
            'e' in C or 'f' in C)])
    return lines

```

This code takes an augmented clique and constructs all possible golden-mean matroids that arise from that augmented clique. The function `all_lines()` then returns all the lines that contain a distinguished clique element `e` and at least one non-clique element. If it returns the empty list, then no matroids were constructed, and the configuration given is forbidden.

6.A.6 Testing For Betsy Ross

```

names = ['a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i'] # 9
    extensions should be plenty
def is_br(n, exts): # n is a positive integer, exts is a list
    of lists.
    grab = [Matroid(Matrix(GF(19),matroids.CompleteGraphic(n).
        representation().rows()))]
    hold = []
    BR = matroids.named_matroids.BetsyRoss()
    for k in range(len(exts)):
        for M in grab:
            for N in M.linear_extensions(simple=True, element=names[k]
                ], fundamentals=funds, F=exts[k]):
                hold.append(N)
    grab = hold
    hold = []
    if all([0.is_isomorphic(BR) for 0 in grab]):
        return True
    else:
        return False

```

This code returns `True` if all matroids constructed by starting with $M(K_n)$ and extending into all the flats in `exts` are isomorphic to the Betsy Ross

(Figure 2.16), and `False` otherwise.

6.A.7 Sublemma 6.6.19.1

6.A.7.1 $r = 4$

```

F = GF(4, 'a')
a = F.gens()[0]
K = matroids.CompleteGraphic(5)
M = matrix(F, [[1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, a, a
], [0, 1, 0, 0, 1, 1, a, a, 0, 1, a, a^2, 0, 0, 0, 0], [0,
0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0], [0, 0, 0, 1,
0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1]]) # G4

N = matrix(F, [[1, 0, 0, 0, 0, 0, a, a, 0, 1, 1, 1, a^2, a^2, a
, a], [0, 1, 0, 0, 1, 1, a, a, 0, 1, a, a^2, 1, 1, 0, 0],
[0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0], [0, 0, 0,
1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1]]) # hp4

print "G_4"
G = Matroid(M)
E = G.groundset()
CC = G.circuit_closures()[2]
for X in Combinations(E,6):
    A = G.delete(X)
    if A.is_isomorphic(K):
        stars = []
        for l in A.cocircuits():
            if len(l) == 4:
                stars.append(list(l))
        print ['stars', stars]
    exts = []
    for x in X:
        Y = copy(X)
        Y.remove(x)
        for c in CC:
            if x in c:

```

```

        flag = True
        for y in Y:
            if y in c:
                flag = False
        if flag == True or len(c) == 5:
            exts.append(list(c))
    print ['extensions', exts]
    print "\r"

print "HP_4"
HP = Matroid(N)
E = HP.groundset()
CC = HP.circuit_closures()[2]
for X in Combinations(E,6):
    A = HP.delete(X)
    if A.is_isomorphic(K):
        stars = []
        for l in A.cocircuits():
            if len(l) == 4:
                stars.append(list(l))
    print ['stars', stars]
    exts = []
    for x in X:
        Y = copy(X)
        Y.remove(x)
        for c in CC:
            if x in c:
                flag = True
                for y in Y:
                    if y in c:
                        flag = False
                if flag == True or len(c) == 5:
                    exts.append(list(c))
    print ['extensions', exts]
    print "\r"

```

This code outputs, for each spanning clique K in both G_4 and HP_4 , two things. First, it outputs the stars in K , allowing us to create a labelled clique easily. It then outputs the rank-two flats that each non-clique element is contained in, allowing us to easily form the augmented clique.

6.A.7.2 $r = 5$

```

F = GF(4, 'a')
a = F.gens()[0]
b = a^2
K = matroids.CompleteGraphic(6)
M = matrix(F, [[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1,
    1, 1, 1, 1, a, a, a], [0, 1, 0, 0, 0, 1, 1, 1, a, a, a, 0,
    0, 0, 1, a, b, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 1, 0, 0,
    1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0], [0, 0, 0, 1,
    0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0],
    [0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1,
    0, 0, 1]])
N = matrix(F, [[1, 0, 0, 0, 0, 0, 0, 0, a, a, a, 0, 0, 0, 1, 1,
    1, b, b, b, a, a, a], [0, 1, 0, 0, 0, 1, 1, 1, a, a, a, 0,
    0, 0, 1, a, b, 1, 1, 1, 0, 0, 0], [0, 0, 1, 0, 0, 1, 0, 0,
    1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0], [0, 0, 0, 1,
    0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0],
    [0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1,
    0, 0, 1]])

print "G_5"
G = Matroid(M)
E = G.groundset()
CC = G.circuit_closures()[2]
for X in Combinations(E,8):
    A = G.delete(X)
    if A.is_isomorphic(K):
        stars = []
        for l in A.cocircuits():

```

```

    if len(l) == 5:
        stars.append(list(l))
print ['stars', stars]
exts = []
for x in X:
    Y = copy(X)
    Y.remove(x)
    for c in CC:
        if x in c:
            flag = True
            for y in Y:
                if y in c:
                    flag = False
            if flag == True or len(c) == 5:
                exts.append(list(c))
print ['extensions', exts]
print "\r"

print "HP_5"
HP = Matroid(N)
E = HP.groundset()
CC = HP.circuit_closures()[2]
for X in Combinations(E,8):
    A = HP.delete(X)
    if A.is_isomorphic(K):
        stars = []
        for l in A.cocircuits():
            if len(l) == 5:
                stars.append(list(l))
print ['stars', stars]
exts = []
for x in X:
    Y = copy(X)
    Y.remove(x)
    for c in CC:

```

```
    if x in c:
        flag = True
    for y in Y:
        if y in c:
            flag = False
    if flag == True or len(c) == 5:
        exts.append(list(c))
print ['extensions', exts]
print "\r"
```

This code outputs, for each spanning clique K in both G_5 and HP_5 , two things. First, it outputs the stars in K , allowing us to create a labelled clique easily. It then outputs the rank-two flats that each non-clique element is contained in, allowing us to easily form the augmented clique.

Bibliography

- [1] ARCHER, S. *Near Varieties and Extremal Matroids*. PhD thesis, Victoria University of Wellington, 2005.
- [2] BIXBY, R. A simple theorem on 3-connectivity. *Linear Algebra Appl.* 45 (1982), 123–126.
- [3] DIESTEL, R. *Graph theory*, fourth ed., vol. 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, 2010.
- [4] DING, G., OPOROWSKI, B., OXLEY, J., AND VERTIGAN, D. Unavoidable minors of large 3-connected matroids. *J. Combin. Theory Ser. B* 71, 2 (1997), 244–293.
- [5] HALL, R., MAYHEW, D., AND VAN ZWAM, S. The excluded minors for near-regular matroids. *European J. Combin.* 32, 6 (2011), 802–830.
- [6] HELLER, I. On linear systems with integral valued solutions. *Pacific J. Math* 7 (1957), 1351–1364.
- [7] KAHN, J. On the uniqueness of matroid representations over $GF(4)$. *Bull. London Math. Soc.* 20, 1 (1988), 5–10.
- [8] KUNG, J. Combinatorial geometries representable over $GF(3)$ and $GF(q)$. I. The number of points. *Discrete Comput. Geom.* 5, 1 (1990), 83–95.
- [9] KUNG, J., AND OXLEY, J. Combinatorial geometries representable over $GF(3)$ and $GF(q)$. II. Dowling geometries. *Graphs Combin.* 4, 4 (1988), 323–332.

- [10] OXLEY, J. *Matroid theory, second edition*. Oxford University Press, USA, 2011.
- [11] OXLEY, J., VERTIGAN, D., AND WHITTLE, G. On maximum-sized near-regular and $\sqrt[6]{I}$ -matroids. *Graphs and Combinatorics* 14, 2 (1998), 163–179.
- [12] PENDAVINGH, R., AND VAN ZWAM, S. Confinement of matroid representations to subsets of partial fields. *J. Combin. Theory Ser. B* 100, 6 (2010), 510–545.
- [13] PENDAVINGH, R., AND VAN ZWAM, S. Lifts of matroid representations over partial fields. *Journal of Combinatorial Theory, Series B* 100, 1 (2010), 36–67.
- [14] PENDAVINGH, R., VAN ZWAM, S., ET AL. *Sage Matroid Package, included in Sage Mathematics Software 6.1.1*. The Sage Matroid Development Team, 2014. <http://www.sagemath.org>.
- [15] SEMPLE, C. Matroid representation over partial fields. Master’s thesis, Victoria University of Wellington, 1995.
- [16] SEMPLE, C. On maximum-sized k -regular matroids. *Graphs Combin.* 15, 4 (1999), 441–462.
- [17] SEMPLE, C., AND WHITTLE, G. Partial fields and matroid representation. *Adv. in Appl. Math.* 17, 2 (1996), 184–208.
- [18] SEYMOUR, P. On minors of non-binary matroids. *Combinatorica* 1, 4 (1981), 387–394.
- [19] STEIN, W., ET AL. *Sage Mathematics Software (Version 6.1.1)*. The Sage Development Team, 2014. <http://www.sagemath.org>.
- [20] VAN ZWAM, S. *Partial Fields in Matroid Theory*. PhD thesis, Technische Universiteit Eindhoven, 2009.
- [21] WELSH, M. Golden-mean and secret sharing matroids. Master’s thesis, Victoria University of Wellington, 2011.

- [22] WHITTLE, G. A characterization of the matroids representable over $GF(3)$ and the rationals. *Journal of Combinatorial Theory, Series B* 65, 2 (1995), 222–261.
- [23] WHITTLE, G. On matroids representable over $GF(3)$ and other fields. *Trans. Amer. Math. Soc.* 349, 2 (1997), 579–603.
- [24] WHITTLE, G. Recent work in matroid representation theory. *Discrete Math.* 302, 1-3 (2005), 285–296.
- [25] WU, Z. On the number of spikes over finite fields. *Discrete Math.* 265, 1-3 (2003), 261–296.
- [26] ZASLAVSKY, T. Voltage-graphic matroids. In *Matroid theory and its applications*. Liguori, Naples, 1982, pp. 417–424.
- [27] ZASLAVSKY, T. Biased graphs. II. The three matroids. *J. Combin. Theory Ser. B* 51, 1 (1991), 46–72.

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