Maximum-Sized Golden-Mean-Graphic Matroids

Maximum-Sized

Definition

Let $\mathcal{M}$ be a minor-closed class of matroids. A rank-r simple matroid M is said to be ***maximum-sized*** in $\mathcal{M}$ if $\forall N \in \mathcal{M}$ such that $r(N) = r$ and N is simple, $|M| \geq |N|$.

Definition

Let $\mathcal{M}$ be a minor-closed class of matroids. The ***growth-rate function*** of $\mathcal{M}$ is the function that defines the number of elements in a maximum-sized matroid of $\mathcal{M}$.

$GF(3)$-representable matroids

Theorem (Whittle, 1997)

Let $\mathcal{F}$ be a set of fields containing $GF(3)$, and let $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. Then for some $q \in \{2, 3, 4, 5, 7, 8\}$, $\mathcal{M}$ is the class of matroids is the class of matroids representable over $GF(3)$ and $GF(q)$.

Historical Results

Maximum-sized results exist for all of Whittle's $GF(3)$-classes of matroids:

- $GF(q)$-representable matroids for each prime power q, in particular $q = 3$.
- Matroids that are representable over $GF(3)$ and $GF(2)$.
- Matroids that are representable over $GF(3)$ and $GF(5)$.
- Matroids that are representable over $GF(3)$ and $GF(4)$.
- Matroids that are representable over $GF(3)$ and $GF(8)$.
- Matroids that are representable over $GF(3)$ and $GF(7)$.

$GF(q)$-representable matroids

Theorem

The matroid $PG(r-1,q)$ is the maximum-sized matroid of rank-r in the class of $GF(q)$-representable matroids. The growth-rate function of the class of $GF(q)$-representable matroids is $\dfrac{q^r-1}{q-1}$.

Regular matroids

Definition

A matroid is ***regular*** if it is representable over $GF(3)$ and $GF(2)$.

Theorem (Heller, 1957)

The maximum-sized regular matroid of rank r is $M(K_{r+1})$. The growth-rate function of regular matroids is $\binom{r+1}{2}$.

Dyadic matroids

Definition

A matroid is ***dyadic*** if it is representable over $GF(3)$ and $GF(5)$.

Theorem (Kung and Oxley, 1988-90)

The maximum-sized dyadic matroid of rank r is $Q_r(GF(3)^*)$, the rank-r ternary Dowling geometry. The growth-rate function of dyadic matroids is r^2.

Near-regular and $\sqrt[6]{1}$ matroids

Definition

A matroid is ***sixth-root-of-unity*** if it is representable over $GF(3)$ and $GF(4)$. A matroid is ***near-regular*** if it is representable over $GF(3)$ and $GF(8)$.

Theorem (Oxley, Vertigan, and Whittle; 1998)

Except for rank 3, both the maximum-sized rank-r near-regular matroid and the maximum-sized rank-r $\sqrt[6]{1}$ *matroid are* T_r^1*. Both classes have growth-rate function* $\binom{r+2}{2} - 2$*.*

$GF(3) \cap GF(7)$-matroids

Theorem

The maximum-sized rank-r matroid representable over $GF(3)$ and $GF(7)$ is $Q_r(GF(3)^*)$. This class has growth-rate function r^2.

$\mathbb{P}$-matrices

Definition

Let $\mathbb{P}$ be a subset of a ring that contains -1 and 0. A $\mathbb{P}$-***matrix*** is a matrix with entries from $\mathbb{P}$ such that every subdeterminant is in $\mathbb{P}$.

Example

The subset $\{-1, 0, 1\}$ of $\mathbb{Z}$ defines the regular matroids.

Definition

Let A be a $\mathbb{P}$-matrix. A subset of k columns is independent if it contains a $k \times k$ subdeterminant that is non-zero. This gives rise to a $\mathbb{P}$-***representable*** matroid.

Definition

The ***golden-mean*** set is $\mathbb{G} = \left\{\pm\tau^i \mid i \in \mathbb{Z}\right\} \cup \{0\}$ where τ is the positive root of $x^2 - x - 1$, also known as the golden ratio.

Golden-mean

Theorem (Semple, Vertigan, Pendavingh, Van Zwam)

Let M be a matroid. The following are equivalent:

(i) *M is representable over both $GF(4)$ and $GF(5)$;*

(ii) *M is golden-mean;*

(iii) *M is representable over $GF(p)$ for all primes p such that $p = 5$ or $p \equiv \pm 1 \pmod 5$, and also over $GF(p^2)$ for all primes p.*

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \tau & 1 & 1 & 0 & 0 & \tau & \tau^2 \\ 0 & 0 & 1 & 1 & \tau^2 & 1 & \tau & -\tau & 1 & 1 & \tau^2 \end{bmatrix}$$

- In this matrix, every non-zero subdeterminant is in the set $\left\{ \pm \tau^i \mid i \in \mathbb{Z} \right\}$.
- For example, $\begin{vmatrix} \tau & \tau \\ \tau^2 & 1 \end{vmatrix} = \tau - \tau^3 = \tau(1 - \tau^2) = \tau(-\tau) = -\tau^2$.

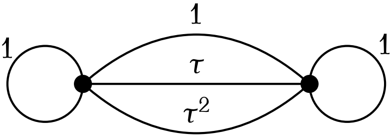
$\mathbb{P}$-graphic

Definition

A matroid M is $\mathbb{P}$-***graphic*** if there exists a $\mathbb{P}$-matrix A with at most two non-zero entries per column such that M is represented over $\mathbb{P}$ by A.

We can scale A so that the first non-zero entry in each column is one.
Note that this naturally gives rise to a (directed) graph with edge labels from $\mathbb{P}$.

Example



$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \tau & \tau^2 \end{bmatrix}$$

A Conjecture and a Theorem

Conjecture (Archer, 2005)

Except for rank 3, the maximum-sized golden-mean matroid of rank-r is one of:

- D_r.
- T_r^2.
- HP_r.

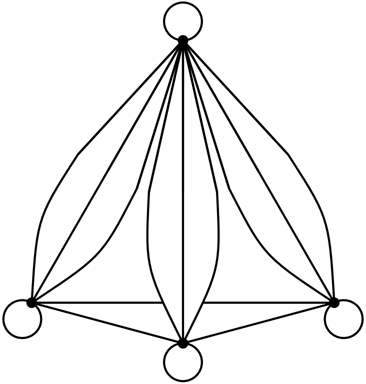
The growth-rate function of golden-mean matroids is $\binom{r+3}{2} - 5$.

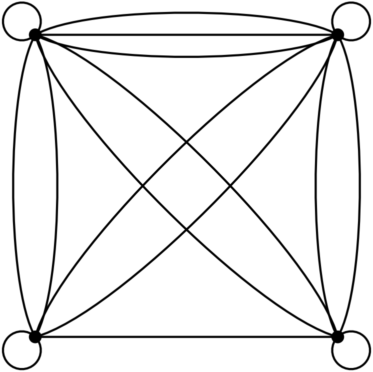
This is too hard, so we restrict to the class of $\mathbb{G}$-graphic matroids.

Theorem (Mayhew, Welsh, 2013)

Archer's conjecture holds for the class of $\mathbb{G}$-graphic matroids.

Pictures





These graphs are directed and have edge weights. However, we are only concerned with their structure, so ignore these.

Results from Archer's Conjecture

Lemma (Welsh)

Let M be a minimum-rank counterexample to Archer's conjecture. Then M is vertically 4-connected.

Lemma (Mayhew, Welsh)

Let M be a minimum-rank counterexample to Archer's conjecture. Then M/e is non-ternary for all $e \in E(M)$.

Proof Outline I

Assume M is a counterexample to Archer's conjecture where $r(M)$ is as small as possible. Note that M is a maximum-sized $\mathbb{G}$-graphic matroid.
Let W be the weighted graph associated with M.

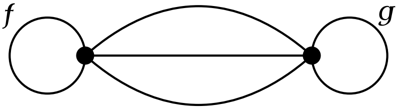
Lemma
W has loops at every vertex.

Lemma
W has no $2K_4$-minor.

Proof Outline II

Lemma

Let e be an element of M. Then M/e is non-ternary. Let f and g be loops in W_e, the weighted graph associated with M/e. Then W_e contains a minor that looks like this:



Proof Outline III

Lemma

In M, a four-point line cannot meet a five-point line.

Lemma

In M, there are no five-point lines.

Proof Outline IV

Lemma

Let X be the subgraph induced by removing all loops and one edge from every parallel class in W. Then X has no K_4-minor.

Lemma

X has a vertex of degree at most two.

Lemma

Let e be a loop in W that is incident with the vertex from the previous lemma. Then M/e is either T^2_{r-1} or D_{r-1}.

Lemma

M does not exist.

Thank You