Mathematical Epidemiology and Acute Rheumatic Fever

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Mathematical Epidemiology

Epidemiology:

- Deals with the incidence and distribution of disease and other health-related states.
- The application of this study to their control.

Experiments in Epidemiology would usually be unethical and unreasonable.

Mathematical models allow us to

- predict when an epidemic might occur.
- predict important features.
- compare control strategies.

Important features of epidemic models

Duration

- How long is the epidemic likely to last?

Time and size of Peak

- The largest number of individuals who will be infected at one time
- How long till we reach that number?

Final Size

- The total number of individuals infected over the course of the epidemic.

Endemic equlibriums

- Is there a stable endemic equilibrium?
- Where does it occur?

Basic Reproduction Number $\mathcal{R}_0$

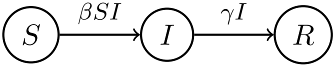
$\mathcal{R}_0$ is the expected number of secondary cases arising from a primary case in a susceptible population. The size $\mathcal{R}_0$ influences

- The size and timing of peaks
- The final size of the epidemic
- If $\mathcal{R}_0 > 1$, a major epidemic is likely.
- If $\mathcal{R}_0 < 1$ then the disease will die out. Only get a minor epidemic.

The SIR epidemic model

- Individuals recover with immunity.
- Divide population into $N = S + I + R$

$$\begin{aligned}\frac{dS}{dt} &= -\beta S I \\ \frac{dI}{dt} &= \beta S I - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

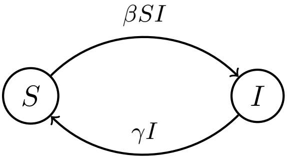


- There is no endemic equilibrium.

The SIS epidemic model

- Individuals recover with no immunity.
- Divide population into $N = S + I$

$$\frac{dS}{dt} = -\beta SI + \gamma I$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$



- The SIS model has an endemic equilibrium at $\beta S = \gamma$.
- This is stable as long as $\beta N > \gamma$.

More on $\mathcal{R}_0$

- For both SIR and SIS, $\mathcal{R}_0 = \frac{\beta}{\gamma}N$.
- For SIS, $\mathcal{R}_0 = 1$ coincides with the point where the endemic equilibrium becomes valid and stable: $\beta S = \gamma$.
- So if $\mathcal{R}_0 < 1$ there can be no endemic equilibrium and the disease will die out.

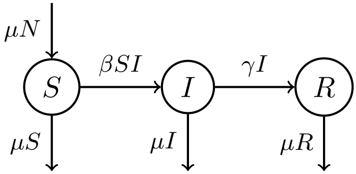
Vital Dynamics

- Vital dynamics cover changes in the population due to births, deaths and migration.
- Whether or not we need to include vital dynamics depends on the time frame of the disease.
  - If we are looking at a one off epidemic the time frame may be too short for births and deaths to make much of an impact.
  - If however, we are looking at an endemic disease, or a series of recurrent epidemics, births and deaths become quite important.

Vital Dynamics

Keeping the population size N constant for simplicity, an SIR model with Vital dynamics would be

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S$$
$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$
$$\frac{dR}{dt} = \gamma I - \mu R$$

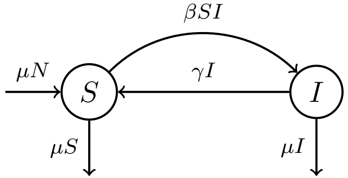


Stochastic Epidemic models

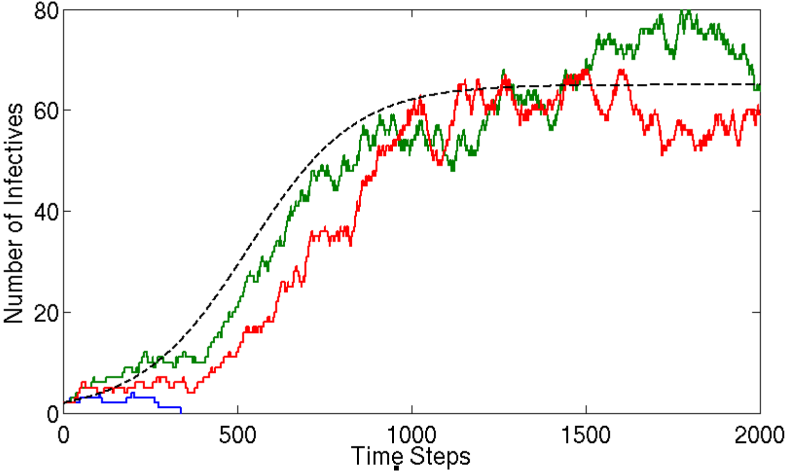
- Using Markov Chains.
- p_{ji} is the probability of one transition from state $I(t) = i$ to state $I(t + \Delta t) = j$ in the time step Δt.
- The transition probabilities are defined using the deterministic model.

For the SIS model with vital dynamics the transition probabilities are:

$$p_{ji}(\Delta t) = \begin{cases} \frac{\beta}{N} i(N-i)\Delta t & j = i+1 \\ (\mu+\gamma) i \Delta t & j = i-1 \\ 1 - [\frac{\beta}{N} i(N-i) + (\mu+\gamma)i]\Delta t & j = i \\ 0 & \text{otherwise} \end{cases}$$



Stochastic Epidemic Models



Acute Rheumatic Fever

Acute Rheumatic Fever (ARF) is an Autoimmune response to a Group A Streptococcus (GAS) infection.

- GAS is commonly recognised as Strep Throat.
- No permanent immunity is retained upon recovery from a GAS infection.
- If a GAS infection is not treated properly, ARF can develop after 2 to 3 weeks.

Acute Rhuematic Fever

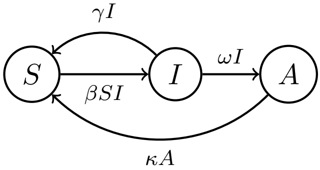
Not everyone with an untreated GAS infection will develop ARF.

- First time attacks of ARF typically occur in children aged between 5 and 17.
- Those who have had ARF before are more likely to get it that those who haven't.
- In NZ Maori and Pacific Island Peoples display ARF rates significantly higher than other ethnicities.

Modelling Acute Rheumatic Fever

A $SIAS$ model with 3 compartments.
Because there is no retained immunity, individuals return to S upon recovery from I or A.

$$\frac{dS}{dt} = -\beta SI + \gamma I + \kappa A$$
$$\frac{dI}{dt} = \beta SI - \gamma I - \omega I$$
$$\frac{dA}{dt} = \omega I - \kappa A$$



This model has an endemic equilibrium at $\beta S = \gamma + \omega$.

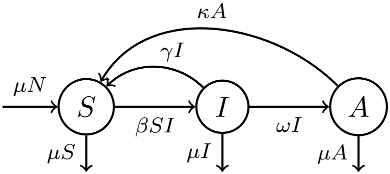
- This is stable as long as $\beta N > \gamma + \omega$

$\mathcal{R}_0 = \frac{\beta N}{\gamma+\omega}$. So if $\mathcal{R}_0 < 1$, there is no stable endemic equilibrium.

Acute Rheumatic Fever with Vital Dynamics

Because ARF is an endemic disease and a long term issue in NZ, vital dynamics are relevant.

$$\frac{dS}{dt} = \mu N - \beta SI + \gamma I + \kappa A - \mu S$$
$$\frac{dI}{dt} = \beta SI - \gamma I - \omega I - \mu I$$
$$\frac{dA}{dt} = \omega I - \kappa A - \mu A$$



The endemic equilibrium is at $\beta S = \gamma + \mu + \omega$.

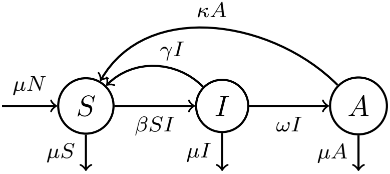
- This is stable for $\beta N > \gamma + \mu + \omega$.

A Stochastic ARF Model

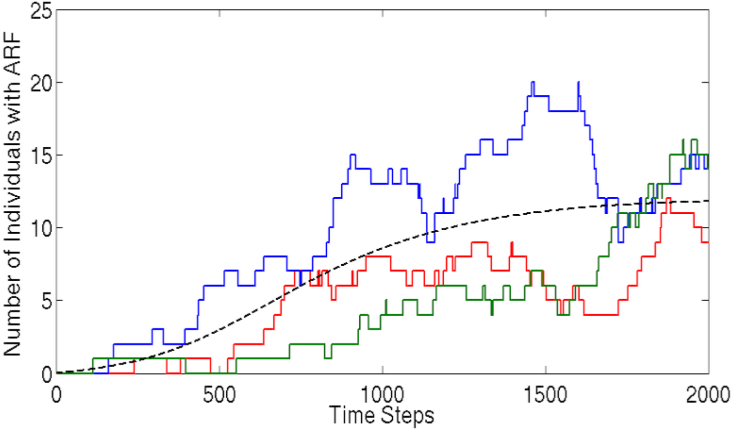
The transition probabilities for the DTMC $SIAS$ model are:

$$p_{(i+j,a+k),(i,a)}(\Delta t) =$$

$$\begin{cases} \frac{\beta i(N-i-a)}{N}\Delta t & (j,k)=(1,0) \\ (\gamma+\mu)i\Delta t & (j,k)=(-1,0) \\ \omega i\Delta t & (j,k)=(-1,1) \\ (\mu+\kappa)a\Delta t & (j,k)=(0,-1) \\ 1-[\frac{\beta i(N-i-a)}{N}+i(\gamma+\omega+\mu)+a(\kappa+\mu)]\Delta t & (j,k)=(0,0) \\ 0 & \text{otherwise} \end{cases}$$



A Stochastic ARF Model



Where to next?

Demographics

- Ethnicity
  - Especially Maori and Pacific Islanders.
- Age Groups
  - Those between 5 and 17 have highest risk.
  - Those over 45 have reduced risk.

ARF and GAS history

- Greater risk for those who have had ARF in the past.