Maximum-Sized Golden-Mean-Graphic Matroids

Maximum-Sized

Definition

Let $\mathcal{M}$ be a minor-closed class of matroids. A rank-r simple matroid M is said to be ***maximum-sized*** in $\mathcal{M}$ if $\forall N \in \mathcal{M}$ such that $r(N) = r$ and N is simple, $|E(M)| \geq |E(N)|$.

Historical Results

Among other classes, historical maximum-sized results exist for:

- $GF(q)$-representable matroids for each prime power q.
- Regular matroids.
- Dyadic matroids.
- Near-regular and $\sqrt[6]{1}$ matroids.
- k-regular matroids.

$GF(q)$-representable matroids

Theorem

The matroid $PG(r-1,q)$ is the maximum-sized matroid of rank-r in the class of $GF(q)$-representable matroids.

Regular matroids

Definition

A matroid is ***regular*** if it is representable over $GF(2)$ and $GF(3)$.

Theorem (Heller, 1957)

The maximum-sized regular matroid of rank r is $M(K_{r+1})$.

Dyadic matroids

Definition

A matroid is ***dyadic*** if it is representable over $GF(3)$ and $GF(5)$.

Theorem (Kung and Oxley, 1988-90)

The maximum-sized dyadic matroid of rank r is $Q_r(GF(3)^*)$, the rank-r ternary Dowling geometry.

Near-regular and $\sqrt[6]{1}$ matroids

Definition

A matroid is ***sixth-root-of-unity*** if it is representable over $GF(3)$ and $GF(4)$. A matroid is ***near-regular*** if it is representable over $GF(3)$, $GF(4)$, and $GF(5)$.

Theorem (Oxley, Vertigan, and Whittle; 1998)

Except for rank 3, both the maximum-sized rank-r near-regular matroid and the maximum-sized rank-r $\sqrt[6]{1}$ matroid are T_r^1.

k-regular matroids

Theorem (Semple, 1999)

The maximum-sized rank-r k-regular matroid is T_r^k.

Partial fields

Definition

A ***partial field*** is a pair (R, G), where R is a commutative ring with unity, and G is a subgroup of the group of units of R such that $-1 \in G$.

Definition

If $\mathbb{P} = (R, G)$ is a partial field, and $p \in R$, then we say that p is an ***element*** of $\mathbb{P}$, denoted $p \in \mathbb{P}$, if $p = 0$ or $p \in G$. Note that if $p, q \in \mathbb{P}$, then $pq \in \mathbb{P}$, but $p + q$ need not be an element of $\mathbb{P}$.

Example

Consider the regular partial field, $\mathbb{U}_0 = (\mathbb{Z}, \{-1, 1\})$. Then $1 \cdot 1 \in \mathbb{U}_0$, but $1 + 1 \notin \mathbb{U}_0$.

$\mathbb{P}$-matrices

Definition

Let $\mathbb{P}$ be a partial field. A $\mathbb{P}$***-matrix*** is a matrix with entries from $\mathbb{P}$ such that every subdeterminant is in $\mathbb{P}$.

Definition

Let A be a $\mathbb{P}$-matrix for some partial field $\mathbb{P}$. A subset of k columns is independent if it contains a $k \times k$ subdeterminant that is non-zero. This gives rise to a $\mathbb{P}$***-representable*** matroid.

Definition

A matroid M is $\mathbb{P}$***-graphic*** for some partial field $\mathbb{P}$ if there exists a $\mathbb{P}$-matrix A with at most two non-zero entries per column such that M is represented over $\mathbb{P}$ by A.

$\mathbb{P}$-graphic

Let M be a $\mathbb{P}$-graphic matroid with associated matrix A.
We can scale A so that the first non-zero entry in each column is one.
Note that this naturally gives rise to a graph with edge labels from $\mathbb{P}$.

Example

Golden-mean

Definition

The ***golden-mean partial-field*** is $\mathbb{G} = (\mathbb{R}, \langle -1, \tau \rangle)$ where τ is the positive root of $x^2 - x - 1$, also known as the golden ratio.

Theorem (Semple, Whittle, Vertigan, Pendavingh, Van Zwam)

A matroid is representable over $\mathbb{G}$ if and only if it is representable over $GF(4)$ and $GF(5)$.

A Conjecture and a Theorem

Conjecture (Archer, 2005)

Except for rank 3, the maximum-sized golden-mean matroid of rank-r is one of:

- D_r.
- T_r^2.
- HP_r.

This is too hard, so we restrict to the class of $\mathbb{G}$-graphic matroids.

Theorem (Mayhew, Welsh, 2013)

Archer's conjecture holds for the class of $\mathbb{G}$-graphic matroids.

Pictures

Results from Archer's Conjecture

Lemma

Let M be a minimum-rank counterexample to Archer's conjecture. Then M is vertically 4-connected.

Lemma

Let M be a minimum-rank counterexample to Archer's conjecture. Then M/e is non-ternary for all $e \in E(M)$.

Proof Outline I

Assume M is a counterexample to Archer's conjecture where $r(M)$ is as small as possible. Note that M be a maximum-sized $\mathbb{G}$-graphic matroid.
Let W be the weighted graph associated with M.

Lemma

W has loops at every vertex.

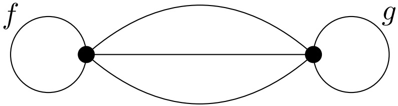
Lemma

W has no $2K_4$-minor.

Proof Outline II

Lemma

Let e be an element of M. Then M/e is non-ternary. Let f and g be loops in W_e, the weighted graph associated with M/e. Then W_e contains a minor that looks like this:



Proof Outline III

Lemma

In M, a four-point line cannot meet a five-point line.

Lemma

In M, there are no five-point lines.

Lemma Proof

Proof Outline IV

Lemma

Let X be the subgraph induced by removing all loops and one edge from every parallel class in W. Then X has no K_4-minor.

Lemma

X has a vertex of degree at most two.

Lemma

Let e be a loop in W that is incident with the vertex from the previous lemma. Then M/e is either T^2_{r-1} or D_{r-1}.

The result follows by case analysis.

Merci