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- Let $\overrightarrow{F} \subseteq \overrightarrow{E}$. Then $\overleftarrow{F} = \left\{\overleftarrow{e} \mid \overrightarrow{e} \in \overrightarrow{F}\right\}$. Note that $\overleftarrow{E} = \overrightarrow{E}$.

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- Let $X, Y \subseteq V$ and $\vec{F} \subseteq \vec{E}$. Now we define $\vec{F}(X, Y) = \left\{(e, x, y) \in \vec{F} \mid x \in X; y \in Y; x \neq y\right\}$.

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- Note that any loops at vertices $v \in X \cap Y$ are ignored by the definitions of $\vec{F}(X, Y)$ and $\vec{F}(v)$.

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  (F2) $f(v, V) = 0$ for all $v \in V$.
- If $X \subseteq V$, note that **(F1)** implies that $f(X, X) = 0$, and that **(F2)** implies that $f(X, V) = 0$.

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Corollary (Gon-far)

If f is a circulation and $e = uv$ is a bridge in G, then $f(e, u, v) = 0$. $\square$

Group-valued Flows

- If f are g are both circulations on G with values in H, then $(f+g)(\vec{e}) = f(\vec{e}) + g(\vec{e})$ and $(-f)(\vec{e}) = -(f(\vec{e}))$ are also circulations. So we have a group.

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- By Corollary Gon-far, a graph with a H-flow cannot have a bridge.

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For every multigraph G there exists a polynomial P such that, for any finite abelian group H, the number of H-flows on G is $P(|H| - 1)$.

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Now assume that there is some edge, $e_0 = uv$ that is not a loop. Let $\vec{e_0} = (e_0, u, v)$ and $E' = E \setminus \{e_0\}$. We consider the multigraphs

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By the induction hypothesis, there are polynomials P_i such that the number of H-flows on G_i is $P_i(k)$, where $k = |H| - 1$. We shall prove that $P_2 - P_1$ is the polynomial we desire.

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We want to show that the H-flows on G_2 are the H-circulations on G that are zero nowhere except possibly on e_0. The set of these circulations on G is denoted F_2. Then $|F_2| = P_2(k)$.
Furthermore, F_2 is the disjoint union of F_1 and F, and hence (1) is true.

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Proof of Theorem: Part 5

Now $f(\overrightarrow{e_0})$ is already determined by **(F2)** for u and the known values of $f(\overrightarrow{e})$ for edges e at u, while $f(\overleftarrow{e_0})$ is already determined by **(F2)** for v and the known values of $f(\overrightarrow{e})$ for edges e at v.

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This polynomial is known as the ***flow polynomial of G***. Like the chromatic polynomial, it is a special case of the Tutte polynomial, the study of which is LTS.

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- If G has no k-flow for any k (K_2 for example), we say $\varphi(G) = \infty$.

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Theorem (Tutte 1950)

A multigraph admits a k-flow if and only if it admits a $\mathbb{Z}_k$-flow.

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A graph is ***even*** if all its vertices have even degree.

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(i) *A graph has a 4-flow if and only if it the union of two even subgraphs.*

(ii) *A cubic graph has a 4-flow if and only if it is 3-edge colourable.*

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Tutte's flow conjectures

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The current pinnacle

Theorem (Seymour 1981)

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Proof.

Exercise. $\square$

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  (F3) $f(\vec{e}) \leq c(\vec{e})$ for all $\vec{e} \in \vec{E}$.
- If the image of f is a subset of $\mathbb{Z}$, then f is ***integral***.

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- By **(F3)** we have $|F| = f(S, \overline{S}) \leq c(S, \overline{S})$. Hence the total value of a flow in N is never larger than the smallest capacity of a cut.
- This bound is always met by some flow.

Ford & Fulkerson 1956

Theorem (Ford & Fulkerson 1956)

In every network, the maximum total value of a flow equals the minimum capacity of a cut.

Proof.

LTS □