On Maximum-Sized Golden-Mean Matroids

Archer's Conjecture

The conjecture we have been trying to prove is due to Archer.

Conjecture

The growth-rate function for the class of golden-mean matroids $\mathcal{G}$ is

$$h_{\mathcal{G}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{G}$ is maximum-sized if and only if M is isomorphic to a member of $\{T_r^2, G_r, HP_r\}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

Two Theorems

Theorem
Let $\mathscr{M}_R$ be the class of golden-mean matroids with an element e such that, upon contracting e, we obtain a regular matroid. Let $\mathscr{R}$ be the class of all minors of members of $\mathscr{M}_R$. Archer's Conjecture is true in $\mathscr{R}$.

Theorem
Let $\mathscr{M}_N$ be the class of golden-mean matroids with an element e such that, upon contracting e, we obtain a near-regular matroid. Let $\mathscr{N}$ be the class of all minors of members of $\mathscr{M}_N$. Archer's Conjecture is true in $\mathscr{N}$.

Another Theorem

Theorem

Let $\mathcal{T}$ be the class of golden-mean graphic matroids. Archer's Conjecture is true in $\mathcal{T}$.

Final Theorem

Theorem

Let $\mathbb{M}_C$ be the class of golden-mean matroids with a spanning clique. Let $\mathcal{C}$ be the class of all minors of members of $\mathbb{M}_C$. Archer's Conjecture holds in $\mathcal{C}$.

Geelen and Nelson result

It is anticipated that our final theorem, when combined with the following forthcoming theorem of Geelen and Nelson, will give a proof of Archer's Conjecture for matroids of sufficiently large rank.

Theorem (Geelen and Nelson)

Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids and let $p(x)$ be a real quadratic polynomial with positive leading coefficient. If $h_{\mathcal{M}}(n) > p(n)$ for infinitely many $n \in \mathbb{Z}^+$, then for all $r, s \in \mathbb{Z}^+$ there exists $M \in \mathcal{M}$ satisfying $\epsilon(M) > p(r(M))$ and $r(M) \geq r$ such that either

(1) *M has a spanning clique restriction, or*

(2) *M is vertically s-connected and there is an s-element independent set S of M so that $\epsilon(M) - \epsilon(M/e) > p(r(M)) - p(r(M) - 1)$ for each $e \in S$.*

Technical Lemmas

Lemma

Let $k \in \mathbb{Z}^+$, let M be a matroid and let N be a minor of M such that $\mathcal{T}_k(N)$ is a tangle. If $X \subseteq E(M)$ is contained in a $\mathcal{T}_k(M, N)$-small set, then there is a minor M' of M such that $M'|X = M|X$, M' has N as a minor, and X is contained in a $\mathcal{T}_k(M', N)$-small set X' such that $E(M') = E(N) \cup X'$ and $\lambda_{M'}(X') = r_{\mathcal{T}_k(M', N)}(X) = r_{\mathcal{T}_k(M,N)}(X)$.

Lemma

There is a function $f : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for all $m, n, l, t \in \mathbb{Z}$ with $m > t \geq 0$, $l \geq 2$, and $n \geq f(m, l)$, if $M \in \mathcal{U}(l)$ has an $M(K_{n+1})$-minor N with corresponding tangle $\mathcal{T} = \mathcal{T}_{\lceil 2n/3 \rceil}(M, N)$ and $X \subseteq E(M)$ satisfies $r_{\mathcal{T}}(X) = t$, then M has a minor M' with an $M(K_{m+1})$-restriction R so that $X \cap E(R) = \emptyset$, $M'|X = M|X$, $E(M') = E(R) \cup X$ and $\lambda_{M'}(X) = t$.