Maximum-Sized Golden-Mean Matroids

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Matroids – Definitions

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- A **circuit**, C, is a subset of E such that C is not independent, but every proper subset of C is.
- A matroid is **simple** if all circuits have size at least 3.

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A matrix over $GF(2)$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

$E = \{a, b, c, d\}$
$\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{c\},$
$\{a, c\}, \{b, c\}\}$

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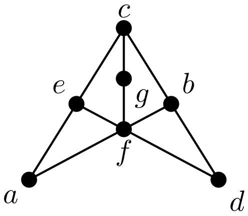
- A set of 3 points in the geometric representation are collinear if and only if they are not independent.

Matroids – An example

A matrix over $GF(4)$

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & \alpha \end{bmatrix}$$



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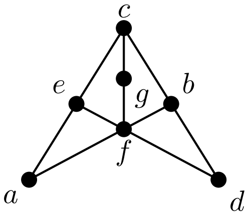
- $\{a, b, c\}$ is independent



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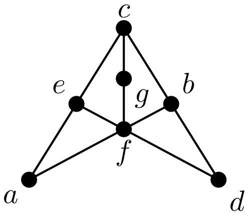
- $\{a, b, c\}$ is independent
- $\{b, c, d\}$ is a circuit



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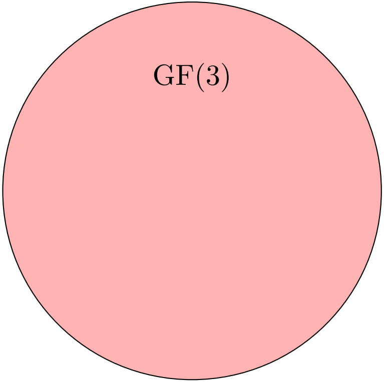
Maximum-Sized

Let $\mathcal{M}$ be a class of simple matroids, and let $M = (E_M, \mathcal{I}_M)$ be a matroid of rank r from $\mathcal{M}$.
M is **maximum-sized** in $\mathcal{M}$ if for every $N = (E_N, \mathcal{I}_N)$ in $\mathcal{M}$ such that N has rank r, $|E_N| \leq |E_M|$.

Characterisation due to Vertigan and Whittle



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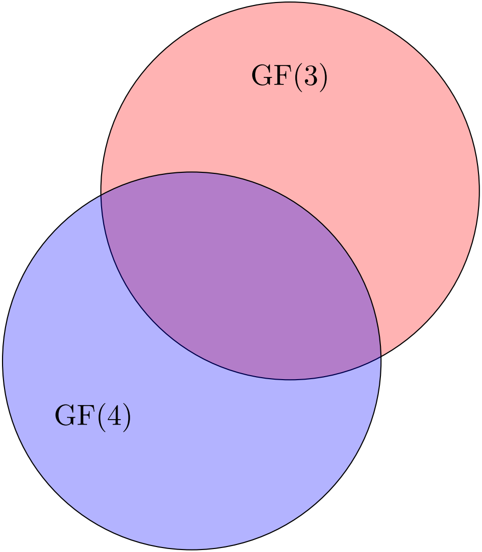

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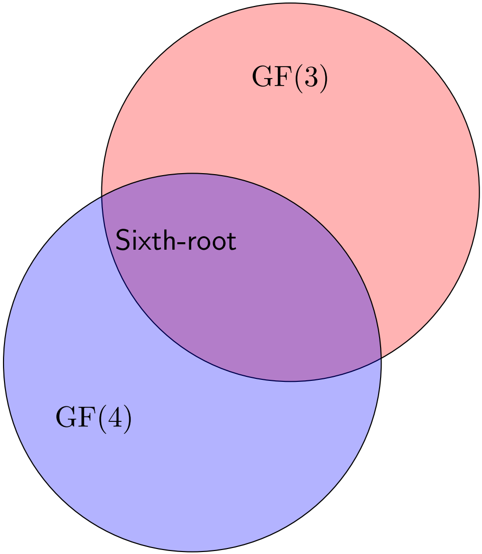

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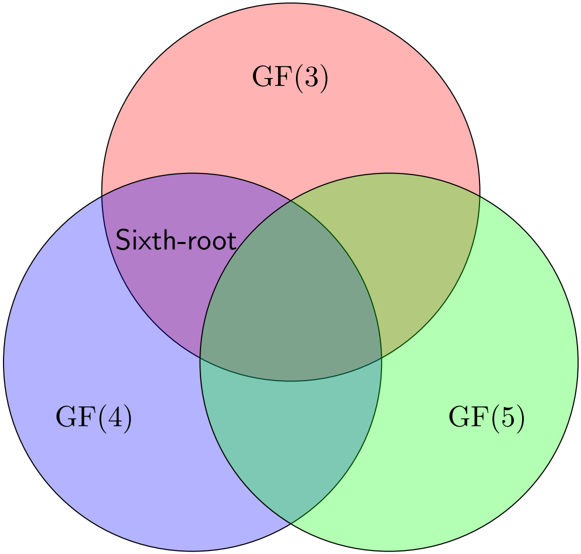

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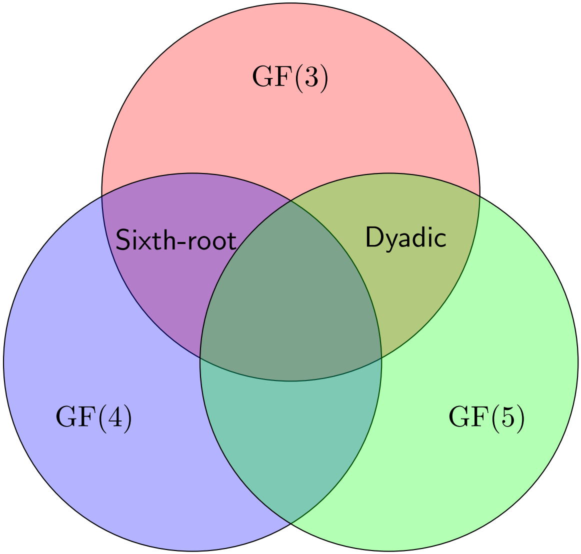

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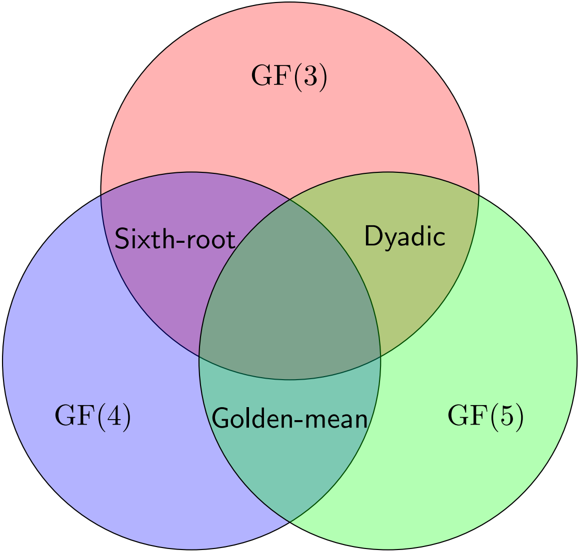

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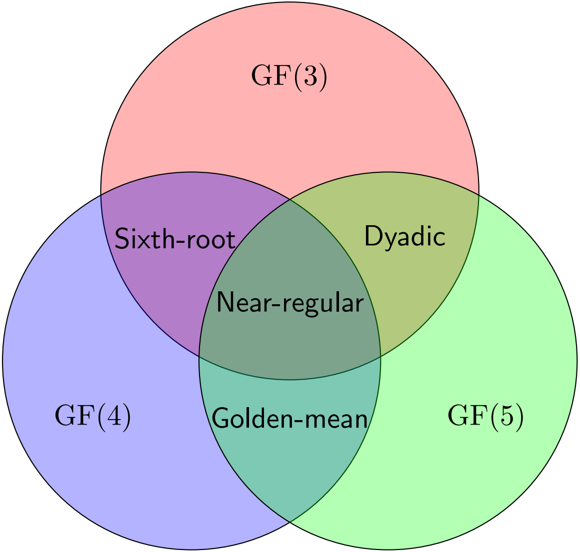

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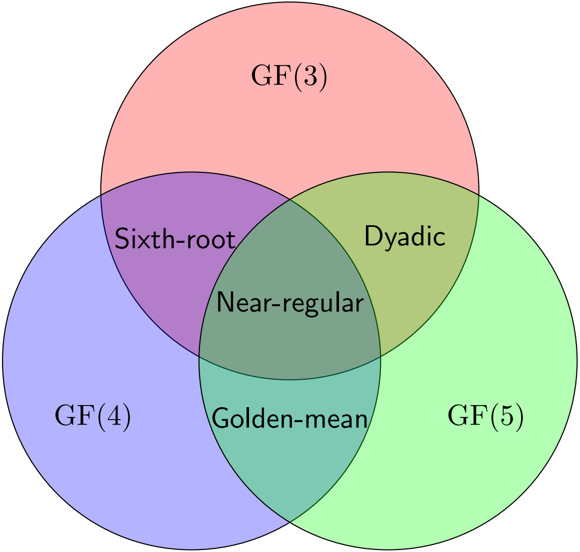

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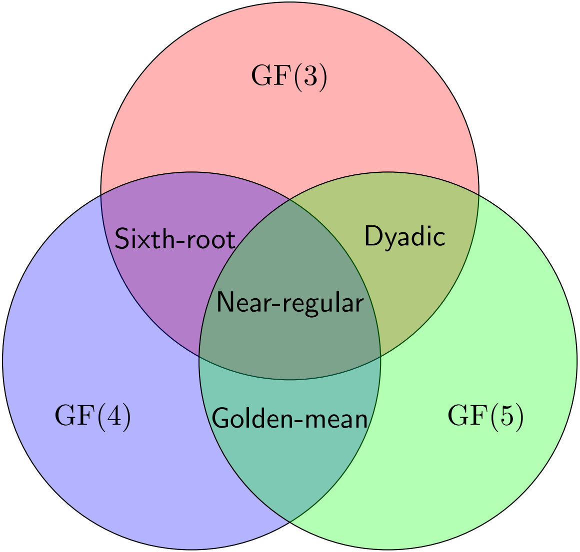

All these classes are also defined by subdeterminants falling into certain sets.

For example, the golden mean determinants are from the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$, where ϕ is the positive root of $\phi^2 - \phi - 1$.

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \phi & 1 & 1 & 0 & 0 & \phi & \phi^2 \\ 0 & 0 & 1 & 1 & \phi^2 & 1 & \phi & -\phi & 1 & 1 & \phi^2 \end{bmatrix}$$

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- In this matrix, every non-zero subdeterminant is in the set $\{\pm\phi^i \mid i \in \mathbb{Z}\}$.
- For example, $\begin{vmatrix} \phi & \phi \\ \phi^2 & 1 \end{vmatrix} = \phi - \phi^3 = \phi(1 - \phi^2) = \phi(-\phi) = -\phi^2$.

The target

Theorem

Let M be a simple maximum-sized golden-mean matroid of rank r. Then M is "some matroid".

Existing Results

The maximum-sized matroids in the following classes have been completely characterised.

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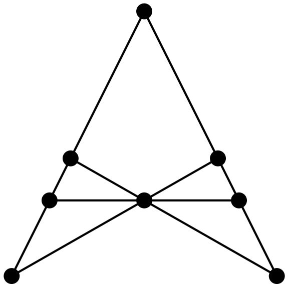
- Sixth-root (Oxley, Vertigan and Whittle, 1998)
- Near-regular (Oxley, Vertigan and Whittle, 1998)

An infinite family – T

$$T_r = \left[\begin{array}{c|ccc|ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & \\ \vdots & & I_{r-1} & & & I_{r-1} & & & I_{r-1} & & & D_{r-1} & \\ 0 & & & & & & & & & & & & \end{array}\right]$$

T_3



Near Regular

Theorem (Oxley, Vertigan and Whittle, 1998)

Let M be a maximum-sized near-regular simple matroid of rank r. Then M is isomorphic to T_r.

Previous Work

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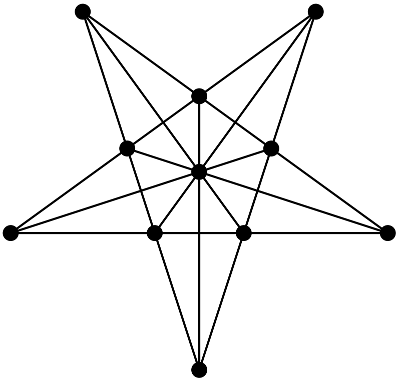
- Semple did some introductory work in his Masters thesis.
- Archer explored low rank golden-mean matroids in his PhD thesis.
- Archer made a conjecture about maximum-sized golden-mean matroids.
- The existing techniques don't work on this conjecture.

Betsy Ross (B_{11})



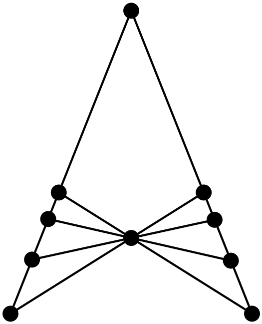
The Betsy Ross matroid, or B_{11}.

An infinite family – T^2

$$T_r^2 = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 \cdots 0 & 1 \cdots 1 & \alpha \cdots \alpha & \alpha^2 \cdots \alpha^2 & 0 \cdots 0 \\ \hline 0 & & & & & \\ \vdots & I_k & I_k & I_k & I_k & D_k \\ 0 & & & & & \end{array}\right]$$

Note that $k = r - 1$.

T_3^2

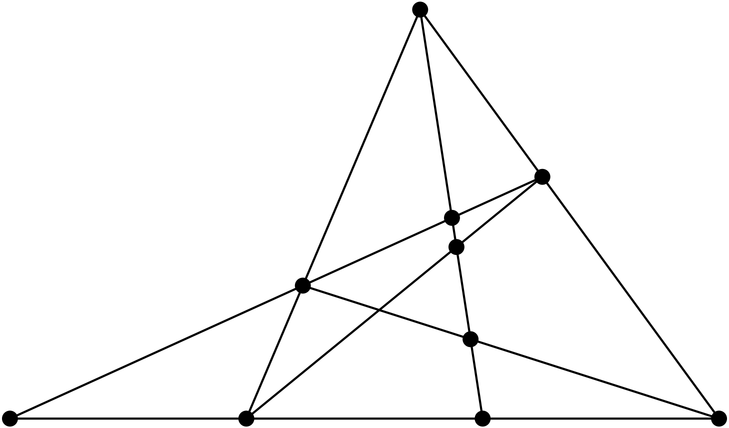


An infinite family - GI

$$GI_r = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc}
-\alpha & -\alpha & -\alpha & 0 & \cdots & 0 & \alpha & \cdots & \alpha & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \alpha & \alpha^2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha & \cdots & \alpha & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\hline
\multicolumn{3}{c|}{0_3^k} & \multicolumn{3}{c|}{I_k} & \multicolumn{3}{c|}{I_k} & \multicolumn{3}{c|}{I_k^0} & \multicolumn{3}{c|}{I_k} & \multicolumn{3}{c|}{I_k^0} & \multicolumn{3}{c}{D_k}
\end{array}\right]$$

Note that $k = r - 2$.

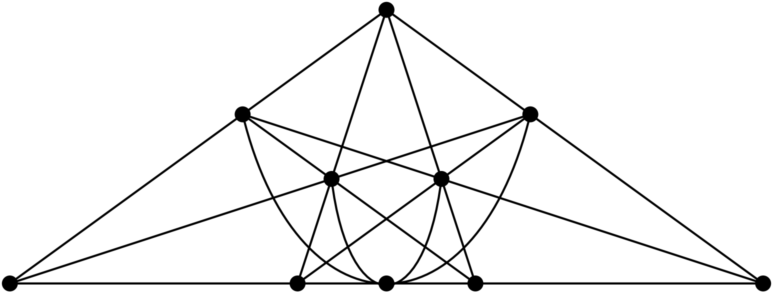
GI_3



An infinite family - GP

$$GP_r = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0 & \cdots & 0\ -1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \phi & \cdots & \phi & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0\ \ \phi & 0 & \cdots & 0 & \phi & \cdots & \phi & \phi & \cdots & \phi & \phi^2 & \cdots & \phi^2 & 0 & \cdots & 0 \\
\hline
 & I_k^0 & & & I_k^0 & & & I_k^0 & & & I_k^0 & & & I_k^0 & & & D_k &
\end{array}\right]$$

GP_3



The Conjecture

Conjecture (Archer, 2005)

Let M be a maximum-sized golden-mean matroid of rank r. If M has rank 3, then M is isomorphic to B_{11}, otherwise M is isomorphic to either GI_r, GP_r, or T_r^2.

Minors

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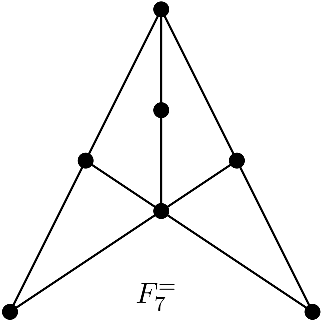
- There are two operations used to remove an element from a matroid:
  1. Deletion; and
  2. Contraction.
- When these operations are used, a **minor** is created.

A stepping stone

Theorem (Welsh, 2010)

Let M be a maximum-sized golden-mean matroid of rank r, with no $F_7^=$ minor. Then M is isomorphic to T_r^2.



$F_7^=$